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Assembly of Fissionable Material in the Presence of a Weak Neutron Source*

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The probability distribution in time at which the neutron population is a slightly supercritical system attains a prescribed level is considered for the case where a source injects well under one neutron per neutron lifetime. For the case of ramp insertion of reactivity it is shown that the energy released in the consequent burst of fissions may in some cases (e.g., unmoderated enriched uranium systems) exceed by over a factor of one hundred the energy release predicted by the reactor kinetics equations.

I. INTRODUCTION

The early growth of neutron population, $n(t)$, within a supercritical system of fissile material is of a statistical nature and may depart significantly from the average time dependence, $\bar{n}(t)$, of an ensemble of these systems. After the initial growth period, the time dependence of neutron population becomes governed by a kinetics equation which, by virtue of the transitory existence of the supercritical system, is essentially nonlinear. Qualitatively, $\bar{n}(t)$ is or is not an approximate solution of the same kinetics equation depending on whether the supercritical configuration is prepared in the presence of a "strong" or "weak" neutron source. When operated in pulsed fashion, Godiva (1) furnishes one example of the weak source case: here, highly enriched uranium metal pieces are brought rapidly to a configuration a few cents above prompt critical with only the spontaneous fission source present. A burst of fissions results which is reproducible in number of fissions ($\sim 10^{16}$) and in width ($\sim 100 \mu\text{sec}$ at half maximum power) but not reproducible as to the time after assembly for the occurrence of peak fission rate. The average time to maximum power following the step increase in reactivity is $\sim 3 \text{ sec}$ or $\sim 3 \times 10^4$ half-widths, and $\bar{n}(t)$, rather than describing the typical growth and decay of the neutron population during a fission pulse, gives primarily a measure of the probability of a fission pulse maximum at time t .

In another "weak" source example, such as the inadvertent assembly of large quantities of fissionable material, energy release as well as time delay will not be reproducible. In this case, where we assume a ramp increase of reactivity, the average number of fissions produced will exceed the value computed under the neglect of fluctuations in neutron population.

For the weak source case and the limited excess reactivity often associated with criticality accidents, simple approximate expressions for the probability distributions in time of burst occurrence and energy release will be obtained. Essentially, the approximation consists in identifying as the main source of fluctuations the distribution between neutron induced finite fission chains and neutron induced persistent (nonfinite) fission chains. One then neglects the additional fluctuations in the growth of neutron populations associated with the latter class. The main sections of this report utilize this approximation while the appendices indicate the range of validity. Section 2 derives an expression for the probability, W , of a source neutron sponsoring a persistent fission chain and defines a "weak" source. Section 3 develops expressions for the probability in time of occurrence of fission bursts following step and ramp increases of reactivity and, for the ramp increase, the probability distribution in energy release associated with the Energy Model (2) of reactivity quenching. Appendices 1 and 2 treat briefly the neutron position and velocity dependence

* Work performed under the auspices of the U. S. Atomic Energy Commission.

of W and the distribution of nonpersistent or finite fission chain lengths while appendices 3 and 4 treat the fluctuations in the growth of persistent chains for special cases of zero and one delayed neutron groups.

II. THE PROBABILITY, W , OF A SOURCE NEUTRON SPONSORING A PERSISTENT FISSION CHAIN AND THE DEFINITION OF A WEAK SOURCE S

We consider a simple reactive system in which all neutrons behave identically, each having the probability p of producing a fission, and each fission having the probability $P(\nu)$ of emitting ν neutrons. Since the probability $(1 - W_f)$ of a source fission not sponsoring a persistent fission chain is equal to the probability that none of its emitted neutrons sponsor such a chain, one has

$$(1 - W_f) = \sum_{\nu=0} P(\nu)[1 - pW_f]^\nu$$

The distribution $P(\nu)$ is known for the common fissionable materials (3) and permits evaluation of Eq. (1) for $W = pW_f$ in terms of the reactivity index ρ .¹ For a slightly supercritical system where $W \ll 1$, one may approximate $[1 - pW_f]^\nu$ as $1 - \nu pW_f + \nu(\nu - 1)p^2W_f^2/2$ and obtain

$$(1 - W_f) = 1 - pW_f\bar{\nu} + \frac{p^2W_f^2}{2} \frac{\bar{\nu}(\bar{\nu} - 1)}{\nu(\nu - 1)} \quad (2)$$

$W \ll 1$

where $\bar{\nu} = \sum_{\nu=0} \nu P(\nu)$ is the average value of the number of neutrons emitted per fission, etc. The quantity $p\bar{\nu}$, which is equal to the average number of daughter fissions produced per fission is conventionally called the reproduction number k . Adopting this notation together with $\Gamma_2 = \frac{\bar{\nu}(\bar{\nu} - 1)}{\nu(\nu - 1)}$ (the notation of reference 3 where it is seen that $\Gamma_2 \approx 0.8$ for the various fissionable nuclides), Eq. (2) may be re-expressed (4):

$$pW_f = W = 2\Delta k / \bar{\nu}\Gamma_2 \quad \text{for } 0 < \Delta k = k - 1 \ll 1 \quad (3)$$

$$W = 0 \quad \text{for } \Delta k < 0$$

To show that Eq. (3) has a more general validity

¹ Equation (1) actually gives W as a double-valued function of ρ , one branch being $W_1(\rho) = 0$, the second having the behavior $W_2(\rho) < 0$ for $\rho < \rho_c$ and $W_2(\rho) > 0$ for $\rho > \rho_c$. The value $\rho_c = 1/\sum_{\nu=0} \nu P(\nu)$ at which these two branches cross is obtained by partially differentiating both sides of Eq. (1) with respect to W and setting $W = 0$. For a subcritical system (i.e., $\rho < \rho_c$) one must reject the $W_2(\rho) < 0$ branch because of the probability interpretation of W , whereas for a supercritical system (i.e., $\rho > \rho_c$) one must reject the $W_1(\rho) = 0$ branch, since this branch denies the existence of persistent chains.

than suggested by the restrictive premises given in the first paragraph, Appendix I develops the analogue of this equation for the probability $W(\mathbf{r}, \mathbf{v}, \Delta k)$ of a neutron, with space-velocity coordinates \mathbf{r}, \mathbf{v} , sponsoring a persistent fission chain.

With a neutron source of strength S neutrons per second, the expected number of persistent chain initiations per second is evidently WS and the average time interval between successive initiations $1/WS$. The expected growth in neutron population $n_i(t)$ associated with the i th persistent chain is of the form $A_i e^{\Delta k(t-t_i)/\tau}$ where τ is the mean neutron lifetime (e folding time multiplied by Δk). The expected value of $n_{i+1}(t)$ is then $e^{-\Delta k/W S \tau} n_i(t)$, and, if $\Delta k/WS\tau \gg 1$, the total population at time t is essentially comprised of neutrons associated with the first persistent fission chain. Using Eq. (3) to eliminate W , this inequality becomes $2S\tau/\bar{\nu}\Gamma_2 \ll 1$ and leads to the definition

$$\text{Weak source: } 2S\tau/\bar{\nu}\Gamma_2 \ll 1 \quad (4)$$

One may note that for a strong source, $S\tau \gg 1$, the neutron population at a time after assembly $t \gg \tau/\Delta k$ is primarily derived from source neutrons occurring in the first e folding time $t_1 = \tau/\Delta k$. During the interval t_1 , the expected number of persistent fission chains sponsored is $\bar{N} = WS t_1 = 2S\tau/\bar{\nu}\Gamma_2$, and the probability distribution in N is Poisson, so that $(\bar{N}^2 - \bar{N})/\bar{N}^2 = 1/\bar{N}$; the fluctuation in neutron population, $n(t)$, is accordingly $[\bar{n}^2(t) - \bar{n}^2(t)]/\bar{n}^2(t) \sim \bar{\nu}\Gamma_2/2S\tau$, or the fluctuation in time at which the population reaches some moderately large prescribed value is given by $(\Delta k)^2(\bar{t}^2 - \bar{t})/\tau^2 \sim \bar{\nu}\Gamma_2/2S\tau$. In the weak source case, to which our attention will be confined, fluctuations in time to a prescribed power are even larger than suggested by the above $1/S$ proportionality.

III. PROBABILITY DISTRIBUTION IN TIME OF OCCURRENCE OF FISSION BURST FOLLOWING STEP AND RAMP INCREASES OF REACTIVITY

Reckoning the zero of time as the instant the system becomes critical, one may consider the time, t , of the occurrence of the fission burst as composed of two parts: $t = t_1 + t_2$, where t_1 represents the time at which the first persistent fission chain is sponsored; and t_2 , the time for the neutron popula-

² Here we have neglected the fluctuations in neutron population buildup associated with a single persistent fission chain. If one designates the asymptotic form of this population as $A_p e^{at}$, the fractional mean-square deviation in total population is more properly

$$[\bar{n}^2(t) - \bar{n}^2(t)]/\bar{n}^2(t) \approx [(\bar{A}_p^2 - \bar{A}_p^2)/\bar{A}_p^2 + 1]/\bar{N} \sim [(\bar{A}_p^2 - \bar{A}_p^2)/\bar{A}_p^2 + 1]\bar{\nu}\Gamma_2/2S\tau$$

tion associated with other square deviations

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A. Step $P(t_1)dt_1$ sponsored denote no probability up to t_1 chain sponsored

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is thus $(S\tau)^2 \ll$ source ca $\bar{t}_2 \approx t_0$ fiducial p where A_0 time t at one has t burst pe $\bar{t} \approx \bar{\nu}\Gamma_2$

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B. Ra layed neu

tion associated with this chain to grow to a maximum or other fiducial value. The average and mean square deviation of t then become formally expressed as

$$\bar{t} = \bar{t}_1 + \bar{t}_2 \quad (5)$$

$$(\bar{t}^2 - \bar{t}^2) = (\bar{t}_1^2 - \bar{t}_1^2) + (\bar{t}_2^2 - \bar{t}_2^2) \quad (6)$$

A. *Step Increase of Reactivity.*—The probability, $P(t_1)dt_1$ of the first persistent fission chain being sponsored at t_1 in the interval dt_1 is (again letting S denote neutron source strength) composed of the probability e^{-Wst_1} of no persistent chains sponsored up to t_1 and the probability $WSdt_1$ of a persistent chain sponsored in dt_1

$$P(t_1)dt_1 = e^{-Wst_1}WSdt_1 \quad (7)$$

and leads to

$$\bar{t}_1 = 1/SW \quad \text{and} \quad \bar{t}_1^2 - \bar{t}_1^2 = (1/SW)^2 \quad (8)$$

When delayed neutrons are not involved in the propagation of persistent chains, the mean square deviation $\bar{t}_2^2 - \bar{t}_2^2$ is small compared to that of $\bar{t}_1^2 - \bar{t}_1^2$ thereby implying that the probability distribution in time of burst occurrence is essentially identical to $P(t_1)dt_1$, save for a simple time translation of \bar{t}_2 . Appendix II develops an expression for the probability $\Phi(n, t)$ of a neutron population n at time t with the boundary condition $\Phi(1, 0) = 1$, and shows that

$$(\bar{t}_2^2 - \bar{t}_2^2) \simeq 1.64(\tau/\Delta k)^2$$

The inequality

$$\bar{t}_2^2 - \bar{t}_2^2 \ll \bar{t}_1^2 - \bar{t}_1^2$$

is thus equivalent to $(\tau/\Delta k)^2 \ll (1/WS)^2$ or $(S\tau)^2 \ll 1$ and is automatically satisfied for the weak source case. As indicated in Appendices III and IV, $\bar{t}_2 \simeq t_0 + 0.577 \tau/\Delta k$ where t_0 is defined by the fiducial population value $n_0 \equiv A_p e^{\Delta k t_0/\tau}$. A_p is A/W where $Ae^{\Delta k t/\tau}$ is the expected neutron population at a time t after insertion of one source neutron. Then one has for the average time from step insertion to burst peak (or other fiducial)

$$\bar{t} \simeq \bar{v}\Gamma_2/2S\Delta k + t_0 + 0.577 \tau/\Delta k \quad (5a)$$

$$\simeq \bar{v}\Gamma_2/2S\Delta k + t_0$$

and for the dispersion

$$\bar{t}^2 - \bar{t}^2 \simeq [\bar{v}\Gamma_2/2S\Delta k]^2 + 1.64 (\tau/\Delta k)^2 \quad (6a)$$

$$\simeq [\bar{v}\Gamma_2/2S\Delta k]^2$$

B. *Ramp Increase of Reactivity.*—Neglecting delayed neutrons the kinetics equation

$$d\bar{n}/dt = (\Delta k/\tau)\bar{n} + S \quad (9)$$

has as a solution for the ramp reactivity increase $\Delta k = at$

$$\bar{n}(t) = \exp(at^2/2\tau) \int_{-\infty}^t dt' S \exp(-at'^2/2\tau) \quad (10)$$

$$\simeq S\sqrt{2\pi\tau/a} \exp(at^2/2\tau) \quad (11)$$

for $t \gg \sqrt{2\tau/a}$

As may be shown from Eq. (10), source fissions occurring prior to $t' = 0$ are responsible for one-half of the average asymptotic neutron population given by Eq. (11). This indicates that persistent fission chains sponsored prior to the system reaching critical play an important role in determining $n(t)$. For a time-dependent reproduction number, the quantity pW_f appearing in the right-hand sides of Eqs. (1) and (2) must be generalized slightly to

$$\frac{1}{v} \int_t^\infty k(t')W_f(t')e^{-(t-t')/\tau} dt'/\tau \quad (12)$$

which merely indicates that a neutron emitted by the source fission at time t is absorbed at the later time t' with probability $e^{-(t'-t)/\tau} dt'/\tau$. Equation (2) then becomes, again neglecting terms higher than W^2

$$\frac{dW_f}{dt} = \frac{W_f}{\tau} \left(\frac{\Gamma_2 W_f}{2} - \Delta k \right) \quad (13)$$

or, for $\Delta k(t \rightarrow \infty) > 0$

$$W(t) = \frac{2\tau}{\bar{v}\Gamma_2 \int_t^\infty dt' \exp\left(-\int_t^{t'} \Delta k dt''/\tau\right)} \quad (14)$$

For the ramp, $\Delta k = at$, Eq. (14) reduces to

$$W(t) = \frac{\sqrt{8a\tau/\pi\bar{v}^2\Gamma_2^2} \exp(-at^2/2\tau)}{1 - E_2(\sqrt{a/2\tau}t)} \quad (15)$$

where $E_2(x) = -E_2(-x)$ is the error integral (5).

The probability of the first persistent fission chain being sponsored at t_1 in the interval dt_1 is

$$P(t_1) dt_1 = \exp\left[-\int_{-\infty}^{t_1} W(t')S dt'\right] W(t_1)S dt_1$$

$$= \sqrt{2a\tau S^2/\pi\bar{v}^2\Gamma_2^2} \left[\frac{1 - E_2\left(\sqrt{\frac{a}{2\tau}}t_1\right)}{2} \right]^{(2S\tau/\bar{v}\Gamma_2)-1} \exp(-at_1^2/2\tau) dt_1 \quad (16)$$

Figure 1 graphs $P(t_1)$ vs t_1 for several values of the source strength parameter $2S\tau/\bar{v}\Gamma_2$. It is seen that, for the weak source case, $P(t_1 < 0)$ is very small,

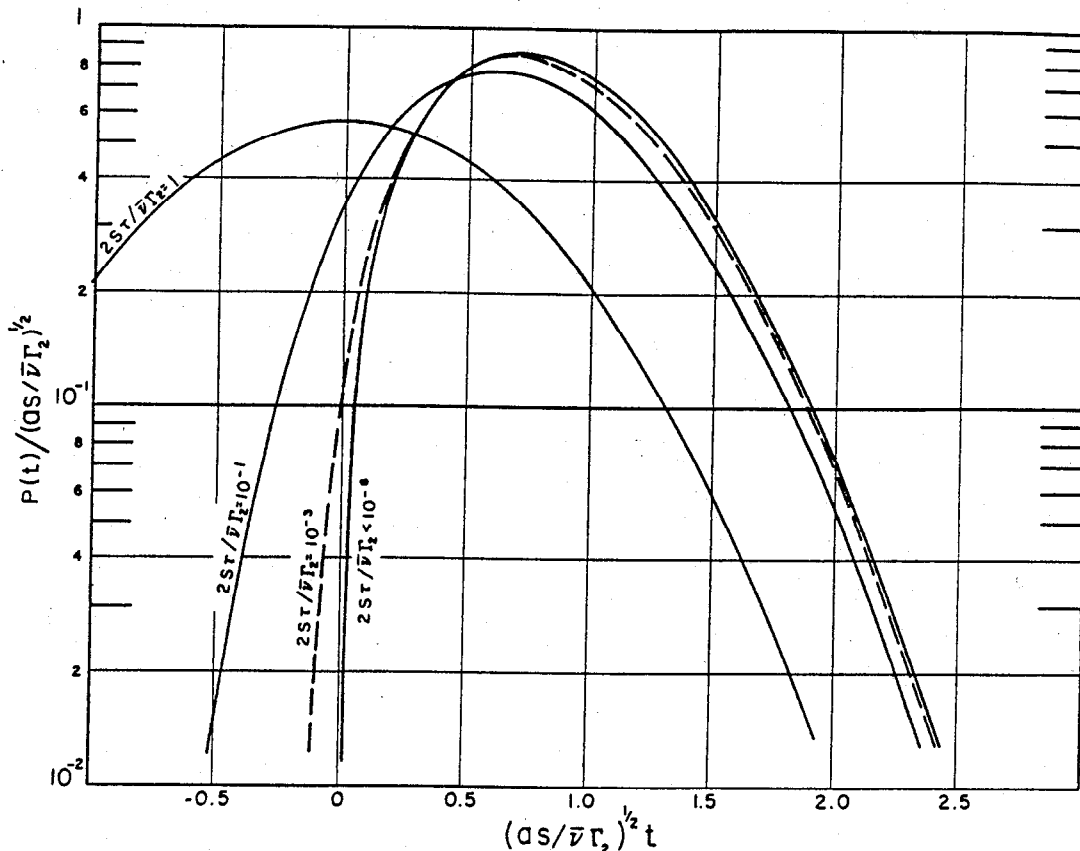


FIG. 1. The probability per unit time, $P(t)$, of the first persistent fission chain being sponsored at time t under the condition of the ramp reactivity insertion $\Delta k = at$ and for several values of the source strength parameter $2S\tau/\bar{v}\Gamma_2$; cf., Eq. (16).

thus indicating that persistent fission chains sponsored prior to the system reaching critical play an unimportant role in determining \bar{l}_1 and thence the average time $\bar{l} = \bar{l}_1 + \bar{l}_2$ [cf., Eq. (5)] for the neutron population to reach a prescribed value. In many types of conceivable criticality accidents (6, 7), the severity is governed by how much excess reactivity, Δk_{max} , is inserted before some overriding quenching mechanism again reduces the reproduction number below unity. For these types of accidents, Δk_{max} is directly dependent on the delay time in fission-power buildup, so that time moments such as \bar{l} or \bar{l}^* are of central importance. An appropriate approximation of Eq. (16) for estimation of these time moments in the weak source case is³

$$P(t) \cong \exp(-aS t_1^2/\bar{v}\Gamma_2) 2aS t_1/\bar{v}\Gamma_2 \text{ for } t_1 \geq 0$$

$$\cong 0 \text{ for } t_1 < 0 \quad (17)$$

³ This approximation is equivalent to representing $W(t)$ by $2\Delta k(t)/\bar{v}\Gamma_2$ for $t > 0$ and by zero for $t \leq 0$, and therefore is not suitable for estimations of $\bar{n}(t)$ or $\bar{n}^2(t)$.

and gives

$$\bar{l}_1 \cong \sqrt{\frac{\pi\bar{v}\Gamma_2}{4aS}} \quad (18)$$

$$\bar{l}_1^2 \cong \frac{\bar{v}\Gamma_2}{aS}$$

The Energy model (2, 7) of reactivity quenching postulates a negative contribution to the total reproduction number proportional to the energy released by fissions, so that with this model and ramp insertion of reactivity, the kinetics equation is

$$\frac{dn}{dt} = \left[at - b \int_{-\infty}^t n(t') dt'/\tau \right] \frac{n}{\tau} + S \quad (9a)$$

where b is a constant. Because Eq. (9a) is nonlinear, the quantity n cannot be identified as the expected neutron population $\bar{n}(t)$; such an identification results, for the weak source case, in the prediction of an energy release, say E_k , less than the correct average, say \bar{E} . To prove this, we review briefly an approximate solution of Eq. (9a): one supposes that

during fission term Eq. (10), the time vanishes, Expressing the energy

$$E_k \cong$$

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⁴ In the equality released by reference the collision of power su

during fission-power buildup the reactivity quenching term is negligible and hence that $n(t)$ is given by Eq. (10). This form of $n(t)$ is presumed to persist to the time t_k when the bracketed term of Eq. (9a) vanishes, after which $n(t)$ rapidly drops to zero. Expressing energy in units of neutron absorptions, the energy released by time t_k is⁴

$$E_k \simeq S\sqrt{2\pi\tau/a} \exp(at_k^2/2\tau)/at_k \simeq at_k/b \quad (19)$$

or (2),

$$bE_k \simeq \sqrt{2a\tau} \ln [2a\tau/(bS\sqrt{2\pi\tau/a})] \quad (19a)$$

To account for the probability distribution in energy release accompanying ramp insertion of reactivity, one identifies as the source of this distribution the probability distribution in occurrence of the first persistent fission chain. Thus, with probability $P(t_1)dt_1$, one has the first persistent chain started at t_1 , the corresponding neutron population,

$$n(t) = \exp [a(t^2 - t_1^2)/2\tau]/W(t_1)$$

and the energy release

$$E(t_1) \simeq [1/W(t_1)at] \exp [a(t^2 - t_1^2)/2\tau] \simeq at/b \quad (20)$$

or

$$bE(t_1) \simeq \sqrt{a^2t_1^2 + 2a\tau} \ln [2a\tau W(t_1)/b] \quad (20a)$$

Comparing Eqs. (19a) and (20a), one finds $E(t_1) > E_k$ when

$$\exp(-at_1^2/2\tau) < (2S\tau/\bar{v}\Gamma_2)\sqrt{2\pi at_1^2/\tau}$$

(18)

From Eq. (17) one then obtains the expected conclusion that the weaker the source strength the more probable the inequality $E(t_1) > E_k$. To illustrate the extent to which the average energy release can exceed the value predicted by Eq. (19a), we consider the ramp insertion of reactivity into Godiva with only the spontaneous fission source present. [This system is especially favorable by virtue of (i) a low effective source strength of about 90 neutrons per second; (ii) a short prompt neutron lifetime of about 0.6×10^{-8} sec (8); and (iii) a known quenching constant $b = 0.5 \times 10^{-19}$ per neutron absorption

⁴ In Eqs. (19) and (19a) we have made use of the inequality $at_k^2/2\tau \gg 1$, which merely implies the energy released by the source fissions is negligible. As is shown in reference 2, an additional energy release of E_k occurs during the collapse of power following t_k . Assuming the ramp insertion of reactivity is terminated immediately after this power surge, the total energy release is $2E_k$.

(9).] Here, from Eqs. (20a) and (18)

$$b\bar{E} = \int bE(t_1)P(t_1) dt_1 \simeq a\bar{t}_1 \simeq \sqrt{\pi a\bar{v}\Gamma_2/4S} \quad (21)$$

which may be rewritten with the use of Eq. (19a) as

$$\bar{E}/E_k \simeq \sqrt{(\pi\bar{v}\Gamma_2/8S\tau)/\ln [2a\tau/(bS\sqrt{2\pi\tau/a})]} \quad (21a)$$

The magnitude of \bar{E}/E_k is primarily determined by the factor $\pi\bar{v}\Gamma_2/8S\tau$ and is insensitive to the values of the ramp parameter, a , and quench parameter, b . For the Godiva example, Eq. (21a) gives $235 \leq \bar{E}/E_k \leq 252$ for $10^{-2} \leq a \leq 10^{-1}$ sec⁻¹. The requirement $a \geq 10^{-2}$ sec⁻¹ insures sufficient penetration into the super-prompt critical region that delayed neutron effects may safely be ignored. For faster insertion rates, the energy model is not strictly applicable to Godiva-like systems (because the thermal expansion which produces reactivity quenching (6, 10) lags the energy generation), and Eq. (21a) underestimates \bar{E}/E_k . Perhaps the main conclusion to be drawn from this numerical example is that criticality accidents can occur, principally with unmoderated enriched uranium, with energy releases exceeding by a couple orders of magnitude the values predicted from a reactor kinetics equation.

APPENDIX 1: THE NEUTRON POSITION-VELOCITY DEPENDENCE OF $W(\mathbf{r}, \mathbf{v})$

Theorem: "A neutron, injected with velocity \mathbf{v} at position \mathbf{r} in a slightly supercritical system, has the probability of sponsoring a persistent fission chain $W(\mathbf{r}, \mathbf{v}) \cong 2\Delta K\phi^+(\mathbf{r}, \mathbf{v})/\bar{v}\Gamma_2$ where $\phi^+(\mathbf{r}, \mathbf{v})$ is the neutron adjoint or effectiveness function for the system normalized by

$$\int d\mathbf{v} d\mathbf{r} \phi(\mathbf{r}, \mathbf{v}) \bar{v}\sigma_f(\mathbf{r}, \mathbf{v}) \left[\int \chi(\mathbf{v}')\phi^+(\mathbf{r}, \mathbf{v}') d\mathbf{v}' \right]^2 = \int d\mathbf{v} d\mathbf{r} \phi(\mathbf{r}, \mathbf{v}) \bar{v}\sigma_f(\mathbf{r}, \mathbf{v}) \times \left[\int \chi(\mathbf{v}')\phi^+(\mathbf{r}, \mathbf{v}') d\mathbf{v}' \right]$$

Here, $\phi(\mathbf{r}, \mathbf{v})$ is the normal mode neutron flux function, σ_f the fission cross section, $\chi(\mathbf{v}')$ the fission neutron spectrum."

A proof of this theorem may be obtained by a procedure paralleling that of the first paragraphs of Section II; that is, obtaining an equation for $W(\mathbf{r}, \mathbf{v})$ by enumerating the contributions, $\Delta_i[1 - W(\mathbf{r}, \mathbf{v})]$, to $[1 - W(\mathbf{r}, \mathbf{v})]$. These contributions are

(1) The probability of the neutron escaping the system without a collision:

$$\Delta_1[1 - W(\mathbf{r}, \mathbf{v})] = 1 - \int_0^\infty \exp\left(-\int_0^s \sigma ds'\right) \sigma ds \quad (1-1)$$

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$$\mathbf{r}' = \mathbf{r} + s\mathbf{v}/v \equiv \mathbf{r} + s\boldsymbol{\Omega}$$

(2) the probability of the neutron being elastically or inelastically scattered at \mathbf{r}' multiplied by the probability that the scattered neutron does not sponsor a persistent fission chain

$$\Delta_2[1 - W(\mathbf{r}, \mathbf{v})] = \int_0^\infty \exp\left(-\int_0^s \sigma ds'\right) \cdot \sigma(\mathbf{v} \rightarrow \mathbf{v}'; \mathbf{r}') [1 - W(\mathbf{r}', \mathbf{v}')] d\mathbf{v}' ds \quad (1-2)$$

(3) the probability of the neutron producing a fission at \mathbf{r}' multiplied by the probability that none of the emitted fission neutrons sponsor a persistent fission chain

$$\Delta_3[1 - W(\mathbf{r}, \mathbf{v})] = \int_0^\infty \exp\left(-\int_0^s \sigma ds'\right) \sigma_f(\mathbf{v}, \mathbf{r}') \sum P(\nu) \cdot \left[\int \chi(\mathbf{v}') d\mathbf{v}' [1 - W(\mathbf{r}', \mathbf{v}')] \right]^\nu ds \quad (1-3)$$

(4) the probability of the neutron being radiatively captured at \mathbf{r}' :

$$\Delta_4[1 - W(\mathbf{r}, \mathbf{v})] = \int_0^\infty \exp\left(-\int_0^s \sigma ds'\right) \sigma_c(\mathbf{r}', \mathbf{v}) ds \quad (1-4)$$

After approximating

$$[1 - W]^\nu \cong 1 - \nu W + [\nu(\nu - 1)/2] W^2,$$

summing these contributions gives

$$W(\mathbf{r}, \mathbf{v}) = \int_0^\infty \exp\left(-\int_0^s \sigma ds'\right) [\sigma(\mathbf{r}', \mathbf{v} \rightarrow \mathbf{v}') + \bar{\nu} \sigma_f(\mathbf{r}', \mathbf{v}) \chi(\mathbf{v}')] W(\mathbf{r}', \mathbf{v}') d\mathbf{v}' ds - \int_0^\infty \exp\left(-\int_0^s \sigma ds'\right) \frac{\Gamma_2 \bar{\nu}^2}{2} \cdot \left[\int \chi(\mathbf{v}') d\mathbf{v}' W(\mathbf{r}', \mathbf{v}') \right]^\nu \sigma_f(\mathbf{r}', \mathbf{v}) ds \quad (1-5)$$

or, the equivalent differential-integral equation

$$[-\boldsymbol{\Omega} \cdot \nabla + \sigma] W(\mathbf{r}, \mathbf{v}) = \int d\mathbf{v}' [\sigma(\mathbf{r}, \mathbf{v} \rightarrow \mathbf{v}') + \bar{\nu} \sigma_f(\mathbf{r}, \mathbf{v}) \chi(\mathbf{v}')] W(\mathbf{r}, \mathbf{v}') - \frac{\Gamma_2}{2} \left[\int d\mathbf{v}' \bar{\nu} \chi(\mathbf{v}') W(\mathbf{r}, \mathbf{v}') \right]^\nu \sigma_f(\mathbf{r}, \mathbf{v}) \quad (1-6)$$

Except for the small term quadratic in W , Eq. (1-6) is recognized as the time independent Boltzmann equation for the neutron flux adjoint function,

$\phi^+(\mathbf{r}, \mathbf{v})$. We thus introduce the normal mode flux equation

$$[\boldsymbol{\Omega} \cdot \nabla + \sigma] \phi(\mathbf{r}, \mathbf{v}) = \int d\mathbf{v}' \left[\sigma(\mathbf{r}, \mathbf{v}' \rightarrow \mathbf{v}) + \frac{\bar{\nu}}{k} \sigma_f(\mathbf{r}, \mathbf{v}') \chi(\mathbf{v}) \right] \phi(\mathbf{r}, \mathbf{v}') \quad (1-7)$$

and, after the conventional procedure of multiplying ϕ equation by W , the W equation by ϕ , forming the difference and integrating over $d\mathbf{r}d\mathbf{v}$, one obtains

$$\left(1 - \frac{1}{k}\right) \int d\mathbf{v} d\mathbf{r} \phi(\mathbf{r}, \mathbf{v}) \bar{\nu} \sigma_f(\mathbf{r}, \mathbf{v}) \int d\mathbf{v}' \chi(\mathbf{v}', \mathbf{r}) W(\mathbf{r}, \mathbf{v}') = \int d\mathbf{v} d\mathbf{r} \phi(\mathbf{r}, \mathbf{v}) \bar{\nu} \sigma_f(\mathbf{r}, \mathbf{v}) \cdot \left[\int d\mathbf{v}' \chi(\mathbf{v}', \mathbf{r}) W(\mathbf{r}, \mathbf{v}') \right]^2 \frac{\bar{\nu} \Gamma_2}{2} \quad (1-8)$$

Since, in the limit $k \rightarrow 1$, Eq. (1-7) becomes adjoint to Eq. (1-6), we have $W(\mathbf{r}, \mathbf{v}) \rightarrow C\phi^+(\mathbf{r}, \mathbf{v})$ where the constant C depends on the normalization of ϕ^+ . After letting

$$\int d\mathbf{v}' \chi(\mathbf{v}', \mathbf{r}) \phi^+(\mathbf{v}', \mathbf{r}) = \phi_0^+(\mathbf{r}),$$

substituting this asymptotic form of $W(\mathbf{r}, \mathbf{v})$ in Eq. (1-8) gives

$$C = \left(1 - \frac{1}{k}\right) \frac{\int d\mathbf{v} d\mathbf{r} \phi(\mathbf{r}, \mathbf{v}) \bar{\nu} \sigma_f(\mathbf{r}, \mathbf{v}) \phi_0^+(\mathbf{r})}{\int d\mathbf{v} d\mathbf{r} \phi(\mathbf{r}, \mathbf{v}) \bar{\nu} \sigma_f(\mathbf{r}, \mathbf{v}) \phi_0^{+2}(\mathbf{r}) \bar{\nu} \Gamma_2 / 2}$$

or

$$W(\mathbf{r}, \mathbf{v}) \cong \left[2 \left(1 - \frac{1}{k}\right) / \bar{\nu} \Gamma_2 \right] \phi^+(\mathbf{r}, \mathbf{v}) \cong 2\Delta k \phi^+(\mathbf{r}, \mathbf{v}) / \bar{\nu} \Gamma_2 \quad (1-9)$$

for the normalization

$$\int d\mathbf{v} d\mathbf{r} \phi(\mathbf{r}, \mathbf{v}) \bar{\nu} \sigma_f(\mathbf{r}, \mathbf{v}) \phi_0^+(\mathbf{r}) = \int d\mathbf{v} d\mathbf{r} \phi(\mathbf{r}, \mathbf{v}) \cdot \bar{\nu} \sigma_f(\mathbf{r}, \mathbf{v}) \phi_0^{+2}(\mathbf{r}).$$

The above equation for C indicates the proper weighting of $\bar{\nu} \Gamma_2$ when this quantity is energy or space dependent. We note that a neutron source with a spectral and spatial distribution

$$\phi(\mathbf{r}, \mathbf{v}) \bar{\nu} \sigma_f(\mathbf{r}, \mathbf{v}) \phi_0^+(\mathbf{r})$$

has the probability, per neutron of sponsoring a persistent fission chain

$$W \cong 2\Delta k / \bar{\nu} \Gamma_2$$

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Especially for an unreflected system, this spatial distribution is centrally peaked more strongly than uniform or normal mode sources, which in turn would then be characterized by average or effective $\phi^+ < 1$. In this report, W is always used in association with a source strength, S , and, rather than belaboring the issue of ϕ^+ , we lump S and ϕ^+ into an effective source strength, $S\phi^+$, still labeled S .

APPENDIX 2: AN ALTERNATIVE DERIVATION OF W THROUGH ENUMERATION OF FINITE CHAIN PROBABILITIES

A specific chain of N fissions (excluding the source fission may be described as follows: the neutrons from a source fission produce N_1 fissions, the neutrons from these N_1 fissions produce N_2 fissions, etc., the total number of fissions being $\sum_i N_i = N$. The probability of a source fission sponsoring a specific chain of N fissions is readily expressed as the product of probabilities of the individual steps $N_i - N_{i+1}$. With our premises of identical neutrons and identical fissions, the probability that the neutrons from the source fission will produce N_1 fissions is

$$\sum_{\nu=N_1}^{\infty} P(\nu) \frac{\nu! p^{N_1} (1-p)^{\nu-N_1}}{N_1! (\nu - N_1)!} \quad (2-1)$$

The probability of the neutrons from these N_1 fissions producing N_2 fissions is

$$\sum_{m=N_2}^{\infty} P(N_1, m) \frac{m! p^{N_2} (1-p)^{m-N_2}}{N_2! (m - N_2)!} \quad (2-2)$$

where $P(N_1, m)$ denotes the probability of N_1 fissions emitting m neutrons and is given by

$$P(N_1, m) = S' \prod_{i=1}^{N_1} P(\nu_i) \quad (2-3)$$

$$\sum_{i=1}^{N_1} \nu_i = m$$

the summation S' being over the partitions $(\nu_1, \nu_2, \dots, \nu_{N_1})$ of m . The probability $W_f(N_1, N_2, \dots, N_g)$ of a specific chain (N_1, N_2, \dots, N_g) of N fissions is thus

$$W_f(N_1, N_2, \dots, N_g) = \prod_{j=1}^{g+1} \sum_{m_j=N_j}^{\infty} P(N_{j-1}, m_j) \frac{m_j! p^{N_j} (1-p)^{m_j-N_j}}{N_j! (m_j - N_j)!} \quad (2-4)$$

where $N_0 = 1$ represents the source fission and $N_{g+1} = 0$. The probability $W_f(N)$ of an N fission chain is then obtained by summing over the possible $W_f(N_1, N_2, \dots, N_g)$ subject to $\sum_{i=1}^g N_i = N$. The summation appears to be most readily accomplished

for neutron number distributions $P(N_{j-1}, m_j)$ whose functional form is independent of N_{j-1} ; eg., the delta distribution $P(\nu) = \delta_{\nu, \bar{\nu}}$ which gives $P(N_{j-1}, m_j) = \delta_{\bar{\nu}, N_{j-1}, m}$, and the Poisson distribution $P(\nu) = \bar{\nu}^\nu e^{-\bar{\nu}} / \nu!$ which gives $P(N_{j-1}, m_j) = [\bar{\nu} N_{j-1}]^{m_j} e^{-\bar{\nu} N_{j-1}} / m_j!$. For these two hypothetical distributions (the delta distribution is of course narrower than the observed distribution of neutron number per fission whereas the Poisson distribution which has $\Gamma_2 = 1$ is somewhat broader) one has

$$W_f(N, \text{delta}) = (1-p)^{\bar{\nu}} [\bar{\nu} p (1-p)^{\bar{\nu}-1}]^N [\bar{\nu} (N+1)]! / (N+1)! [\bar{\nu} (N+1) - N]! \bar{\nu}^N \quad (2-5)$$

$$W_f(N, \text{Poisson}) = e^{-\bar{\nu}^2} [\bar{\nu} p e^{-\bar{\nu}^2}]^N [N+1]^N / (N+1)! \quad (2-6)$$

The probability, W_f , of a source fission sponsoring a persistent fission is now obtained by subtracting from unity the sum $\sum_{N=0}^{\infty} W_f(N)$. This summation is readily accomplished if one notes that the right hand sides of both Eqs. (2-5) and (2-6) are of the form $y(p)x^N(p)A(N)$ where $x(p)$ (either $\bar{\nu} p (1-p)^{\bar{\nu}-1}$ or $\bar{\nu} p e^{-\bar{\nu}^2}$) is such that to each $p \geq 1/\bar{\nu}$ there is a $p^* \leq 1/\bar{\nu}$ for which $x(p^*) = x(p)$. But, for a fission probability $p^* < 1/\bar{\nu}$, there are no persistent chains and

$$\sum_{N=0}^{\infty} y(p^*) x^N(p^*) A(N) = 1$$

or

$$\sum_{N=0}^{\infty} x^N(p^*) A(N) = 1/y(p^*)$$

For a fission probability $p > 1/\bar{\nu}$, then,

$$y(p) \sum_{N=0}^{\infty} x^N(p) A(N) = y(p)/y(p^*)$$

where $p^* < 1/\bar{\nu}$ is defined by $x(p^*) = x(p)$. Thus,

$$W_f(\text{delta}) = 1 - (1-p)^{\bar{\nu}} / (1-p^*)^{\bar{\nu}} \quad (2-7)$$

with $\bar{\nu} p^* (1-p^*)^{\bar{\nu}-1} \equiv \bar{\nu} p (1-p)^{\bar{\nu}-1}$

and $p^* \bar{\nu} \leq 1$

$$W_f(\text{Poisson}) = 1 - e^{-\bar{\nu}(p-p^*)} \quad (2-8)$$

with $\bar{\nu} p^* e^{-\bar{\nu} p^*} \equiv \bar{\nu} p e^{-\bar{\nu} p}$

and $p^* \bar{\nu} \leq 1$

Although Eqs. (2-7) and (2-8) are readily derived from Eq. (1), the explicit introduction of p^* permits the determination of all moments of finite chain

lengths by repeated partial differentiations of W_f : the n th moment, \bar{N}^n , of the finite chain lengths is

$$\begin{aligned} \bar{N}^n &= \sum_{N=0}^{\infty} N^n W_f(N) / \sum_{N=0}^{\infty} W_f(N) \\ &= \sum N^n y(p) x^N A(N) / \sum y x^N A(N) \\ &= [x \partial / \partial x]^n \sum y(p) x^N A(N) / \sum y x^N A(N) \\ &= [1 - W_f]^{-1} (x \partial / \partial x)^n [1 - W_f]. \end{aligned}$$

More simply

$$\bar{N}^n(\text{delta}) = (1 - p^*)^{\bar{v}} \left[\frac{p^*(1 - p^*)}{1 - \bar{v}p^*} \frac{\partial}{\partial p^*} \right]^n (1 - p^*)^{-\bar{v}} \quad (2-9)$$

$$\bar{N}^n(\text{Poisson}) = e^{-\bar{v}p^*} \left[\frac{p^*}{1 - \bar{v}p^*} \frac{\partial}{\partial p^*} \right]^n e^{\bar{v}p^*} \quad (2-10)$$

The n th moments thus depend only on p^* and one has the finite chain length distributions of a supercritical system duplicated in a subcritical system with the same p^* value.

For reproduction numbers $p\bar{v}$ close to unity, Eqs. (2-9) and (2-10) may be approximated as

$$\bar{N}^n \approx \left(\frac{\Gamma_2}{2}\right)^{n-1} \frac{(2n-2)!}{(n-1)!} \left(\frac{1}{|\Delta k|}\right)^{2n-1} \quad (2-11)$$

where $\Gamma_2 = (\bar{v} - 1)/\bar{v}$ and unity for the delta and Poisson distributions, respectively.

Of interest for the discussion of Appendix 4, are the moments \bar{m}^l of the number, m , of delayed neutron precursors produced by the prompt fission chain multiplication of a source neutron in a system whose reactivity lies between delayed and prompt critical. Noting that in the present manner of reckoning fission chain length, the probability $W(N+1)$ of a source neutron sponsoring $N+1$ fissions is $pW_f(N)$, and assuming that the probability of obtaining m precursors from $(N+1)$ fissions is the Poisson value $[\beta\bar{v}(N+1)]^m e^{-\beta\bar{v}(N+1)}/m!$ one has

$$\begin{aligned} \left[\frac{m!}{(m-l)!} \right] &= p \sum_{N=0}^{\infty} \sum_{m=0}^{\infty} W_f(N) [\beta\bar{v}(N+1)]^m \\ &\quad \cdot e^{-\beta\bar{v}(N+1)} / (m-l)! \\ &= p(\beta\bar{v})^l (\bar{N}+1)^l \approx p(\beta\bar{v})^l \\ &\quad \cdot \left(\frac{\Gamma_2}{2}\right)^{l-1} \frac{(2l-2)!}{(l-1)!} \left(\frac{1}{|\Delta k_p|}\right)^{2l-1} \end{aligned} \quad (2-12)$$

or

$$\bar{m}^l \approx \frac{1}{\bar{v}} (\beta\bar{v})^l \left(\frac{\Gamma_2}{2}\right)^{l-1} \frac{(2l-2)!}{(l-1)!} \left(\frac{1}{|\Delta k_p|}\right)^{2l-1} \quad (2-13)$$

The approximation of (2-13) follows from (2-12) because $|\Delta k_p| \ll 1$. As only prompt neutron linked fission chains are included, the prompt reproduction number k_p has been substituted for k in Eq. (2-11).

APPENDIX 3: THE PROBABILITY $\mathcal{O}(n, t)$ OF n NEUTRONS AT TIME t

If the probability for a neutron to be absorbed, including leakage, in the infinitesimal time interval dt is dt/τ , then $\mathcal{O}(n, t)[1 - ndt/\tau]$ represents the probability of n neutrons at time t and no absorptions in the ensuing time interval dt , and thus contributes to the value of $\mathcal{O}(n, t + dt)$. If the probability for a neutron absorption to cause fission is p , then $\mathcal{O}(n+1, t)[(1-p)(n+1)dt/\tau]$ represents the probability of $n+1$ neutrons at time t and one nonfission producing absorption in the ensuing time interval dt ; i.e., a second possibility for having n neutrons at $t + dt$. If fission produces ν neutrons with probability $P(\nu)$, the remaining contribution to $\mathcal{O}(n, t + dt)$ is

$$\sum_{\nu=0}^{\infty} \mathcal{O}(n+1-\nu, t) [(n+1-\nu)P(\nu)pdt/\tau]$$

and one has (11)

$$\begin{aligned} \mathcal{O}(n, t + dt) &= [1 - ndt/\tau]\mathcal{O}(n, t) \\ &\quad + [(n+1)(1-p)dt/\tau]\mathcal{O}(n+1, t) \\ &\quad + \sum_{\nu=0}^{\infty} [(n+1-\nu)P(\nu)pdt/\tau]\mathcal{O}(n+1-\nu, t) \end{aligned}$$

or the equivalent

$$\begin{aligned} \tau \frac{d}{dt} \mathcal{O}(n, t) &= -n\mathcal{O}(n, t) \\ &\quad + (1-p)(n+1)\mathcal{O}(n+1, t) \\ &\quad + p \sum_{\nu=0}^{\infty} P(\nu)(n+1-\nu)\mathcal{O}(n+1-\nu, t) \end{aligned} \quad (3-1)$$

The l th moment of the neutron number distribution

$$\bar{n}^l = \sum n^l \mathcal{O}(n, t),$$

is obtained by multiplying Eq. (3-1) by n^l and summing over n ,

$$\begin{aligned} \tau \frac{d}{dt} \bar{n}^l &= \bar{n}^{l+1} + \sum_{q=0}^l \left\{ (1-p)(-1)^q \right. \\ &\quad \left. + p \sum_{\nu=0}^{\infty} P(\nu)(\nu-1)^q \right\} l! \bar{n}^{l+1-q} / q!(l-q)! \end{aligned}$$

which becomes, after setting

$$(1-p)(-1)^q + p \sum_{\nu=0}^{\infty} P(\nu)(\nu-1)^q = b_q$$

and noting $b_0 = 1$

$$\tau \frac{d}{dt} \bar{n}^l = \sum_{q=1}^l l! b_q n^{l+1-q} / q! (l-q)! \quad (3-2)$$

With $n = 1$ at $t = 0$, and with time independent b_q , the first and second moments, for example, are

$$\begin{aligned} \bar{n} &= e^{b_1 t / \tau} \\ \bar{n}^2 &= \left(1 + \frac{b_2}{b_1}\right) e^{2b_1 t / \tau} - \frac{b_2}{b_1} e^{b_1 t / \tau} \end{aligned} \quad (3-3)$$

where

$$b_1 = p\bar{v} - 1 = k - 1$$

and

$$b_2 = p\bar{v}^2 - 2p\bar{v} + 1 = \bar{v}\Gamma_2 k + 1 - k.$$

For a slightly supercritical system, $b_1 \ll 1$, and at times $b_1 t / \tau \gg 1$, the moments have the simple expression

$$\bar{n}^l = \left[1 - \frac{2\Delta k}{\bar{v}\Gamma_2}\right] \delta_{l,0} + l! \left[\frac{\bar{v}\Gamma_2}{2\Delta k}\right]^{l-1} e^{l\Delta k t / \tau} \quad (3-4)$$

where $\delta_{l,0} = 0$, for $l \neq 0$, $= 1$ for $l = 0$.

The distribution function $\mathcal{P}(n, t)$ associated with Eq. (3-4) is thus, noting the identity

$$\begin{aligned} \int_0^\infty n^l e^{-an} dn &= l! [1/\alpha]^{l+1} \\ \mathcal{P}(n, t) &= \left[1 - \frac{2\Delta k}{\bar{v}\Gamma_2}\right] \delta_{n,0} + 2\Delta k / \bar{v}\Gamma_2 \left[\frac{\bar{v}\Gamma_2}{2\Delta k} \right]^n \\ &\quad \cdot e^{-\Delta k t / \tau} \exp \left[- (2\Delta k n / \bar{v}\Gamma_2) e^{-\Delta k t / \tau} \right] \end{aligned} \quad (3-5)$$

Equation (3-5) is then of the form

$$\mathcal{P}(n, t) = [1 - W] \delta_{n,0} + W \mathcal{P}_p(n, t) \quad (3-6)$$

where W is the probability of the source neutron sponsoring a persistent fission chain and $\mathcal{P}_p(n, t)$ is the probability of n neutrons at time t under the condition of a persistent chain sponsored at $t = 0$.

To show the effect of neglecting the fluctuations in neutron population growths among different persistent fission chains sponsored at $t = 0$, we note that when $\mathcal{P}_p(n, t)$ is summed over a small population range Δn about n_0 , the result, $\mathcal{P}_p(n_0, t) \Delta n$, is interpretable either as the probability that the neutron population is in the range Δn about n_0 at time t or the probability that the neutron population reaches the value n_0 in the time interval $\Delta t = \tau \Delta n / n_0 (k - 1)$ about t . Letting n_0 define the time t_0 through

$$n_0 = [\bar{v}\Gamma_2 / 2(k - 1)] e^{(k-1)t_0 / \tau}$$

we have: If a source neutron sponsors a persistent

fission chain at $t = 0$, then the probability per unit time that the neutron population reaches n_0 is

$$\begin{aligned} \mathcal{P}_p(n_0, t) n_0 (k - 1) / \tau &= \frac{\Delta k}{\tau} e^{-\Delta k(t-t_0) / \tau} \\ &\quad \cdot \exp \left[- e^{-\Delta k(t-t_0) / \tau} \right] \end{aligned} \quad (3-7)$$

Figure 2 graphs $\mathcal{P}_p(n_0, t) n_0$ versus $\Delta k(t - t_0) / \tau$. The most probable time for the neutron population to reach n_0 is t_0 , the average time is $\bar{t} = t_0 + 0.577 \tau / (k - 1)$, and the mean square deviation is

$$\bar{t}^2 - \bar{t}^2 = 1.64 [\tau / (k - 1)]^2$$

These same values apply for a source fission (as contrasted to a source neutron) sponsoring a persistent chain at $t = 0$, as the probability of more than one of the neutrons emitted from the source fission giving rise to an infinite number of progeny is small ($\sim W$). As was seen in Section 3, the condition

$$1.64 [\tau / \Delta k]^2 \ll (1/W S)^2$$

for ignorability of fluctuations in buildup times of persistent chains is automatically satisfied for the weak source case. From Eq. (3-3) one finds the fractional mean square deviation in the neutron

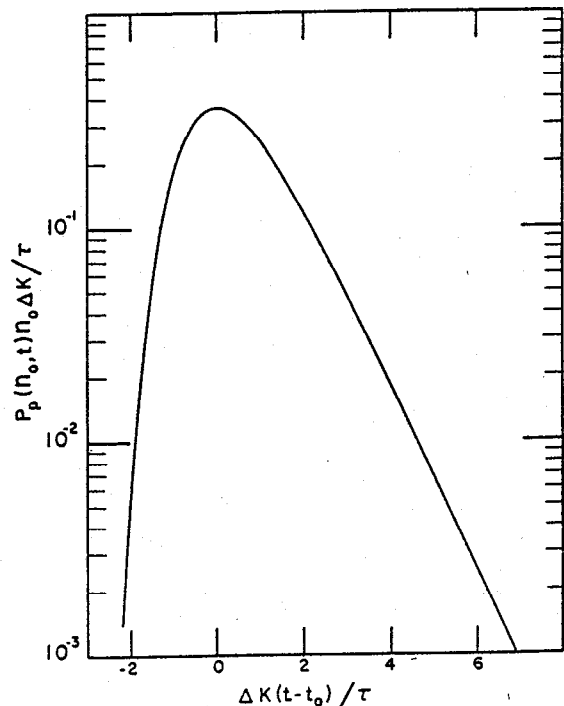


FIG. 2. The probability per unit time, $\mathcal{P}_p(n_0, t) n_0 \Delta k / \tau$, of the neutron population reaching $n_0 = [\bar{v}\Gamma_2 / 2\Delta k] e^{\Delta k t_0 / \tau}$ at a time t after a persistent fission chain is sponsored. Here time is expressed in units of $\tau / \Delta k$; cf., Eq. (3-7).

population attaining at large times the value

$$(\bar{n}^2 - \bar{n}^2)/\bar{n}^2 \approx \bar{\nu}\Gamma_2/(k - 1) = 2/W.$$

For time-dependent reproduction number, the moments \bar{n} and \bar{n}^2 are readily obtained from Eq. (3-2) and give

$$(\bar{n}^2 - \bar{n}^2)/\bar{n}^2 \approx \bar{\nu}\Gamma_2 \int_0^\infty dt' \cdot \exp \left[- \int_0^{t'} (k - 1) dt''/\tau \right] / \tau$$

which, as is seen from Eq. (15) of Section 3, is equal to $2/W(0)$. The analogues of Eqs. (3-4) through (3-7) are obtained by replacing $2\Delta k/\bar{\nu}\Gamma_2$ by $W(0)$ and $\Delta kt/\tau$ by $\int_0^t \Delta k dt'/\tau$.

APPENDIX 4: FLUCTUATIONS IN PRECURSOR POPULATION, C , FOR THE CASE $0 < \Delta k \ll \beta$, $\tau_0 \ll \beta\tau_d \ll 1/S$ (WHERE τ_0 IS THE NEUTRON LIFETIME, τ_d THE PRECURSOR LIFETIME, β THE DELAYED NEUTRON FRACTION FROM FISSION, AND S THE NEUTRON SOURCE)

The reactivity region $0 < \Delta k \ll \beta$, like the region $\Delta k \gg \beta$, is one where the coupling between precursor and neutron populations is relatively innocuous. In the region between delayed and prompt critical ($0 < \Delta k < \beta$), the death of a precursor results, with probability $P(m)$, in m new precursors through the agency of the prompt neutron linked finite fission chain sponsored by the associated delayed neutron. If the neutron lifetime is sufficiently short, these m new precursors may be considered as instantaneously produced and the analysis for the probability $\mathcal{O}(C, t)$ of C precursors at time t becomes equivalent to the development for $\mathcal{O}(N, t)$ given in Appendix 3. Specifically, with the neutron source S ,

$$\tau_d \frac{d}{dt} \mathcal{O}(C, t) = - [C + S\tau_d] \mathcal{O}(C, t) + S\tau_d \sum \mathcal{O}(C - m, t) P(m) + \sum \mathcal{O}(C - m + 1, t) P(m) [C - m + 1]$$

or

$$\tau_d \frac{d}{dt} \bar{C}^l = \sum_{q=1}^l [\bar{C}^{l+1-q} (\bar{m} - 1)^q + S\tau_d \bar{C}^{l-q} \bar{m}^q] \frac{l!}{q!(l-q)!}$$

With the conditions $\bar{C}^l(0) = 0$, and time-independent source and reactivity, the expressions for the first two moments are

$$\bar{C} = \frac{S\tau_d \bar{m}}{(\bar{m} - 1)} [e^{(\bar{m}-1)t/\tau_d} - 1]$$

$$\bar{C}^2 = \frac{S\tau_d}{2(\bar{m} - 1)^2} \{ \bar{m}(\bar{m} - 1)^2 + 2S\tau_d \bar{m}^2 + \bar{m}^2(\bar{m} - 1) \} \cdot \{ e^{2(\bar{m}-1)t/\tau_d} - 1 \} - \frac{S\tau_d}{(\bar{m} - 1)^2} \cdot \{ \bar{m}(\bar{m} - 1)^2 + 2S\tau_d \bar{m}^2 \} \{ e^{(\bar{m}-1)t/\tau_d} - 1 \}$$

Under the restrictions, $\bar{m} - 1 \ll 1$ and $S\tau_d \bar{m} \ll (\bar{m} - 1)^2$, which, as will be seen, are equivalent to $\Delta k \ll \beta$ and $\beta S\tau_d \ll 1$, respectively, the asymptotic (in time) expressions for the moments become

$$\bar{C}^l = \frac{S\tau_d}{(\bar{m} - 1)} \left[\frac{(\bar{m} - 1)^2}{2(\bar{m} - 1)} \right]^{l-1} (l - 1)! e^{l(\bar{m}-1)t/\tau_d} \quad l \neq 0$$

and yield the asymptotic probability distribution

$$\mathcal{O}(C, t) = \int_0^\infty WS dt' e^{-Wst'} \mathcal{O}_p(C, t - t') \quad (4-6)$$

where

$$W = 2(\bar{m} - 1)/(\bar{m} - 1)^2$$

TABLE I
TIME FOR GODIVA POWER TO BUILD UP TO 2.7×10^{11} FISSIONS/SEC FOLLOWING A LARGE STEP INCREASE OF REACTIVITY TO $k - 1 = 0.0047_6$ ($\Delta k/\beta = 0.70$)^{a, b}

Run number	Time (sec)
1	33.1
2	30.5
3	30.1
4	32.4
5	34.2
6	31.9
7	31.4
8	37.8
9	31.2
10	33.8
11	26.3
12	25.8
13	34.3
14	35.8
15	43.9
16	28.6
17	25.4
18	39.3
19	26.7
20	31.5
21	27.0
22	27.8

^a The neutron source is $S \approx 90$ neutrons/sec and the stable positive period is $1/\alpha \approx 1.76$ sec. (corresponding to $2S\tau/\bar{\nu}\Gamma_2 = 2S\Delta k/\bar{\nu}\Gamma_2 \alpha \approx 0.75 < 1$).

^b Average time: $\bar{t} = 31.8$ sec. Mean square deviation:

$$\bar{t}^2 - \bar{t}^2 = 21 \text{ sec}^2.$$

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$$\Phi_p(C, t - t') = W e^{-(\bar{m}-1)(t-t')/\tau_d} \cdot \exp[-W C e^{-(\bar{m}-1)(t-t')/\tau_d}]$$

From Appendix 2, one has

$$(\bar{m} - 1) = (\beta/|\Delta k_p| - 1) \simeq \Delta k/\beta$$

and

$$\overline{(\bar{m} - 1)^2} = \bar{\nu} \Gamma_2 \beta^2 / |\Delta k_p|^2 \simeq \bar{\nu} \Gamma_2 / \beta$$

Thus W assumes the familiar value $2\Delta k/\bar{\nu} \Gamma_2$ and $\Phi_p(C, t - t')$ is identical to the distribution $\Phi(N, t - t')$ of Appendix 3 save for the replacement of neutron lifetime τ_0 by the effective lifetime or cycle time $\beta\tau_d$. Here again, fluctuations in the buildup of persistent chains is ignorable in the weak source case

$$2S\beta\tau_d/\bar{\nu} \Gamma_2 \ll 1$$

Unfortunately the restriction

$$\bar{m} - 1 \simeq \Delta k/\beta \ll 1$$

is much more limiting than the inequality $\Delta k \ll 1$ which applies in the absence of delayed neutrons. An indication that fluctuations in the buildup of persistent chains may not be ignorable for $\Delta k \sim \beta$ (i.e., near prompt-critical) in the weak source case is given by Eqs. (4-3) and (4-4). These equations give for the asymptotic value of the fractional mean square deviation in precursor population

$$\frac{\bar{C}^2 - \bar{C}^2}{\bar{C}^2} = \frac{\bar{\nu} \Gamma_2}{4|\Delta k_p| \tau_d S} \left[1 + \left(\frac{4\beta}{|\Delta k_p|} - 3 \right) \right] \quad (4-7)$$

$$0 < -\Delta k_p \leq \beta$$

Here, the parenthesized quantity represents the fractional mean-square deviation in persistent populations (see footnote 2) and is seen to deviate markedly from unity as prompt critical is approached. (The singularity at $\Delta k_p = 0$ is, of course, incorrect and is associated with the assumption employed in Eq. (4-1) that the precursors formed by prompt neutron linked fission chains are instantaneously produced.) Similarly, from the precursor C_i and neutron n population moments equations given by Courant and Wallace (11), one finds for the condition $\Delta k_p \gg (\beta\tau_0/\tau_d)^{1/2}$, $\Phi(n, t = 0) = \delta_{n,1}$

$$\frac{\bar{n}^2 - \bar{n}^2}{\bar{n}^2} = \frac{1}{W} \left[1 + \left(1 + \frac{2\beta}{|\Delta k_p|} \right) \right] \quad (4-8)$$

$$\Delta k_p \gg (\beta\tau_0/\tau_d)^{1/2}$$

For fast systems, with neutron lifetimes of the order of 10^{-8} sec, both Eqs. (4-7) and (4-8) indicate considerable fluctuation in population growth near prompt critical. Experimental data on fission rate

buildup in Godiva (with only the spontaneous fission source present) for $\Delta k/\beta \simeq 1.05$ (1) indicate that $\{\bar{t}_2^2 - \bar{t}_2^2\}^{1/2}$ is comparable to $(1/WS)$. Additional data on the fission rate buildup at $\Delta k/\beta = 0.7$ are summarized in Table I. The average time and mean square deviation in time after the reactivity step increase for the power to build up to 2.7×10^{11} fissions/sec are predicted by Eq. (5a) and Eq. (6a) to be $\bar{t} \simeq 32.5$ sec and $\bar{t}^2 - \bar{t}^2 \simeq 10.9$ sec² [the arbitrary fiducial of 2.7×10^{11} fissions/sec enters only in the computation of t_0 : this power level corresponds, in Godiva, to the neutron population $n_0 = 4.2 \times 10^3$; the value of \bar{A} is determined from a solution of the Kinetics equation; e.g., $\bar{A} \simeq \alpha\beta\tau_0/(\Delta k[\beta - \Delta k]) \simeq 2.4 \times 10^{-6}$ and yields $\bar{A}_p \simeq 5.1 \times 10^{-4}$; thence $t_0 \simeq 28.1$ sec]. Thus again Eq. (6a), applied in the vicinity of prompt critical, accounts for about one-half of the observed time fluctuation ($\bar{t}^2 - \bar{t}^2$) indicating that here the growth of a persistent chain is considerably more erratic than anticipated for either $\Delta k \ll \beta$ or $\Delta k \gg \beta$.

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