ON THE MINKOWSKI MEASURE

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ABSTRACT. The Minkowski Question Mark function relates the continued-fraction representation of the real numbers to their binary expansion. This function is peculiar in many ways; one is that its derivative is 'singular'. One can show by classical techniques that its derivative must vanish on all rationals. Since the Question Mark itself is continuous, one concludes that the derivative must be non-zero on the irrationals, and is thus a discontinuous-everywhere function. This derivative is the subject of this essay.

Various results are presented here: First, a simple but formal measure-theoretic construction of the derivative is given, making it clear that it has a very concrete existence as a Lebesgue-Stieltjes measure, and thus is safe to manipulate in various familiar ways. Next, an exact result is given, expressing the measure as an infinite product of piece-wise continuous functions, with each piece being a Möbius transform of the form $(ax+b)/(cx+d)$. This construction is then shown to be the Haar measure of a certain transfer operator. A general proof is given that any transfer operator can be understood to be nothing more nor less than a push-forward on a Banach space; such push-forwards induce an invariant measure, the Haar measure, of which the Minkowski measure can serve as a prototypical example. Some minor notes pertaining to it's relation to the Gauss-Kuzmin-Wirsing operator are made.

1. INTRODUCTION

The Minkowski Question Mark function was introduced by Hermann Minkowski in 1904 as a continuous mapping from the quadratic irrationals to the rational numbers[23]. This mapping was based on the consideration of Lagrange's criterion, that a continued fraction is periodic if and only if it corresponds to a solution of a quadratic equation with rational coefficients. Minkowski's presentation is brief, but is based on a deep exploration of continued fractions. Minkowski does provide a graph of the function; a modern version of this graph is shown in figure 1.

Arnaud Denjoy gave a detailed study of the function in 1938[6], giving both a simple recursive definition, as well as expressing it as a summation. Its connection to the continued fractions allowed him to provide explicit expressions for its self-similarity, described as the action of Möbius transformations. Although Denjoy explores a generalization that is defined for the non-negative real number line, this essay will stick to Minkowski's original definition, defined on the unit interval. Little is lost by sticking to the original form, while keeping the exposition simpler.

For rational numbers, Denjoy's summation is written as

$$?(x) = 2\sum_{k=1}^{N} (-1)^{k+1}\ 2^{-(a_1+a_2+\ldots+a_k)} \tag{1.1}$$

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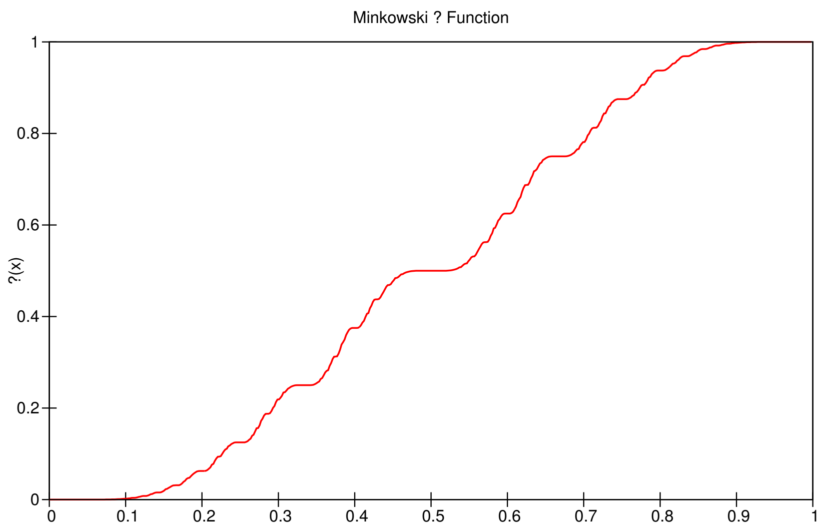
FIGURE 1.1. The Minkowski Question Mark Function



where the a_k are the integers appearing in the continued fraction[18] expansion of x:

$$x = [a_0; a_1, a_2, \cdots, a_N] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

For the case of general, irrational x, one simply takes the limit $N \to \infty$; it is not hard to show that the resulting function is well-defined and continuous in the classical sense of delta-epsilon limits.

The function is symmetric in that $?(1-x) = 1 - ?(x)$, and it has a fractal self-similarity which is generated by [27]

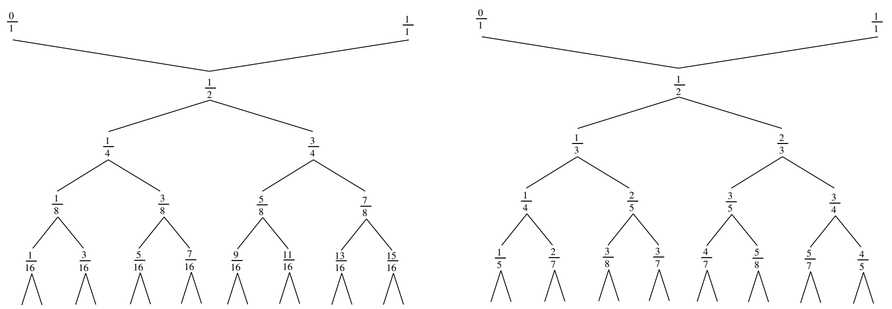
$$\frac{1}{2} ?(x) = ?\left(\frac{x}{1+x}\right) \tag{1.2}$$

An equivalent, alternative definition of the function can be given by stating that it is a correspondence between rationals appearing in the Stern-Brocot, or Farey tree, and the binary tree of dyadic rationals, shown in figure 1; it maps the Farey tree into the dyadic tree.

Many insights into the nature and structure of the function, including self-similarity and transformation properties, can be obtained by contemplating the repercussions of this infinite binary-tree representation. Most important is perhaps the insight into topology: the presence of the tree indicates a product topology, and guarantees that the Cantor set will manifest itself in various ways.

The Question Mark has many peculiarities, and one is that it's derivative is "singular" in an unexpected way[25]. One can show, by classical techniques, that its derivative must vanish on all rationals: it is a very very flat function, as it approaches any rational. Since the Question Mark itself is continuous, one concludes that the derivative must be non-zero, and infinite, on the irrationals, and is thus a discontinuous-everywhere function. The

FIGURE 1.2. Dyadic and Stern-Brocot Trees



derivative of the Minkowski Question Mark function is interpretable as a measure, and is often called the 'Minkowski measure'. It is this measure that is the subject of this essay.

The derivative has been a subject of recent study. Dushistova, *et al.* show that the derivative vanishes not only on the rationals, but also on a certain class of irrationals[9]; this refines a result from Paridis[24]. Kesseböhmer computes the Hausdorff dimension of the sets on which the deriviative is non-zero[17].

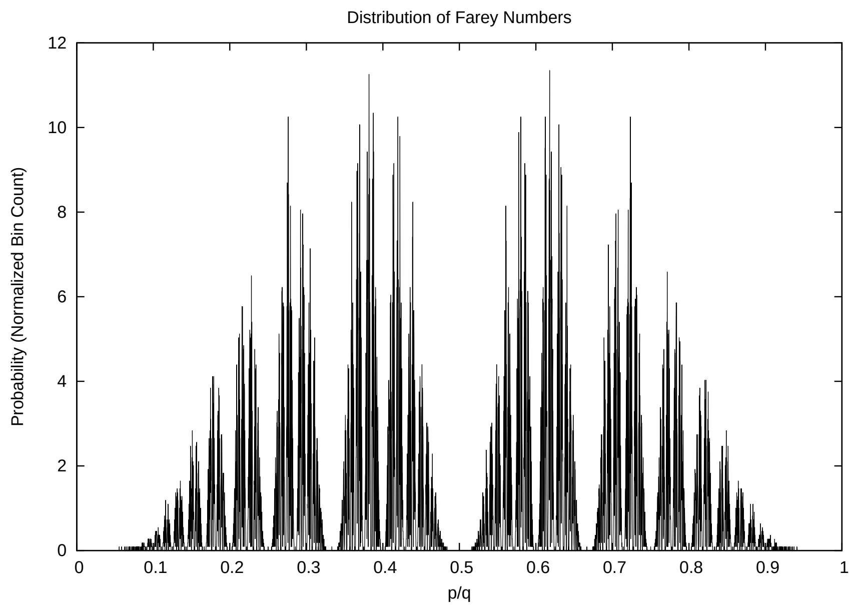
The remainder of this text is structured as follows: the next section introduces the derivative, or 'Minkowski measure', as the statistical distribution of the Farey numbers over the unit interval. The third section reviews the lattice-model or cylinder-set topology for binary strings, while the fourth presents an exact result for the Minkowski measure in this topology. This results disproves a conjecture by Mayer[22] that the distribution is given by the Kac model[16]. In the fifth section, the language of lattice models is converted back to the standard topology on the real number line, to give an exact result for the measure as an infinite product of piece-wise C^∞ functions, with each piece taking the form of a Möbius transform $(ax+b)/(cx+d)$. The sixth section provides a short note on modularity; the seventh reviews an application, the eighth some generalizations. The nineth section introduces the transfer operator, the tenth section provides an abstract theoretical framework, demonstrating that the transfer operator should be regarded as a pushforward. This is found to induce an invariant measure, the Haar measure. The next section provides a grounding for the the previous abstract discussions, illustrating each theoretical construction with its Minkowski measure analog. The next two sections review a twisted form of the Bernoulli operator, and the Gauss-Kuzmin-Wirsing operator, respectively. The papaer concludes with a call to develop machinery to better understand the discrete eigenvalue spectrum of these operators.

2. THE MINKOWSKI MEASURE

It can be shown that the Stern-Brocot or Farey tree enumerates all of the rationals in the unit interval exactly once[26, 5]. By considering this tree as a statistical experiment, as the source of values for a random variable, one obtains a statistical distribution; this distribution is exactly the Minkowski measure.

The statistical experiment may be run as follows: consider the first N rows of the tree: these provide a pool of 2^N rationals. A random variable X_N is then defined by picking randomly from this pool, with equal probability. Ultimately, the goal is to consider the random variable $X = \lim_{N\to\infty} X_N$, and understand its distribution. From the measure-theory

FIGURE 2.1. Distribution of Farey Fractions



This figure shows a histogram of the first $2^N = 65536$ Farey fractions according to their location on the unit interval. To create this figure, the unit interval is divided up into $M = 6000$ bins, and the bin count c_m for each bin m is incremented whenever a fraction p/q falls within the bin: $(m-1)/M \leq p/q < m/M$. At the end of binning, the count is normalized by multiplying each bin count c_n by $M/2^N$. The normalization ensures that the histogram is of measure one on the unit interval: that $\sum_{m=0}^{M} c_m = 1$.
The shape of this histogram is dependent on the choice of M and N. However, it does exactly capture the results of performing an experiment on the random variable X_N and tallying the results into evenly-spaced bins. The result is 'exact' in the sense of eqn 2.1: the height of each bin is precisely given by the differences of the value of ? at its edges, up to the granularity imposed by N. As $N \to \infty$, the errors are uniformly bounded and vanishing.

perspective, one is interested in the the probability Pr that a measurement of X will fall in the interval $[a,b]$ of the real number line. The distribution is the measure $\mu(x)$ underlying this probability:

$$\Pr[a \leq X < b] = \int_a^b d\mu = \int_a^b \mu(x)\, dx$$

It is straightforward to perform this experiment, for finite N, on a computer, and to graph the resulting distribution. This is shown in figure 2.1. What is most notable about this figure is perhaps that it clearly belongs to a class of functions called 'multi-fractal measures'[12, 21].

The measure μ is known as the 'Minkowski measure'; its relationship to the Question Mark function is straight-forward, and may be stated as a formally:

Theorem. *The integral of the Stern-Brocot distribution is the Minkowski Question Mark function; that is,*

$$?(b) - ?(a) = \int_a^b d\mu \tag{2.1}$$

Proof. The result follows from the duality of the ordinary dyadic tree and the Stern-Brocot tree. Consider a different statistical experiment, performed much as before, but where this time, one picks fractions from the dyadic tree. Call the resulting random variable Y, understood as the limit of discrete variables Y_N with $N \to \infty$. Since the dyadic rationals are uniformly distributed on the unit interval, the corresponding measure is trivial: its just the uniform distribution. Yet this 'trivial' experiment differs from that giving X only by the labels on the tree. Performing the same experiment, in the same way, but just with different labellings of the values, is simply a mapping from one set to another; the mapping is already given by eqn 1.1, it is $?(x)$. The transformation of measures under the action of maps is a basic result from probability theory; the relationship between the random variables is simply that $Y_N = ?(X_N)$ and $Y = ?(X)$. The measure, of course, transforms inversely, and so by standard measure theory, one has

$$\Pr[a \leq X < b] = ?(b) - ?(a)$$

essentially, concluding the proof of the assertion 2.1. □

Because the equation 2.1 resembles a classical integral, one is sorely tempted to 'abuse the notation', and use the classical notation to write:

$$\frac{d?(x)}{dx} = ?'(x) = \mu(x) \tag{2.2}$$

The use of the derivative notation in the above is rather misleading: properly speaking, the derivative of the Question Mark is not defined when using the standard topology on the real number line. However, the treatment of μ as a measure can be very rigorously founded. Specifically, μ can be seen to be an example of an outer measure; by the measure extension theorem (see, for example [20]), an outer measure can be extended to a unique measure on the unit interval of the real number line.

This is done as follows. Let $2 = \{0,1\}$ be the set containing two elements; let 2^ω be the (semi-)infinite product space $2 \times 2 \times 2 \times \cdots$. This space consists of all (infinitely-long) strings in two letters. An outer measure is a set function $\nu : 2^\omega \to [0,\infty]$ that is monotone (*i.e.* $\nu(A) < \nu(B)$ whenever $A \subset B \subset 2^\omega$), that is sigma-subadditive (*i.e.* given any countable set $A_1, A_2, \ldots$ of subsets of 2^ω and $A \subset \bigcup_{k=0}^{\infty} A_k$, then $\nu(A) \leq \sum_{k=0}^{\infty} \nu(A_k)$) and that ν of the empty set is zero. It is straight-forward to show that $\nu = ?' \circ ?^{-1}$ satisfies each of these properties for an outer measure; this essentially follows from the total ordering of the fractions on the Stern-Brocot tree (as well as the total ordering of the dyadic rationals on the dyadic tree). The remaining step is to notice that the "natural" topology on 2^ω is the so-called "weak topology" (sometimes called the "cylinder set topology"), and that the weak topology is finer than the natural topology on the real number line. That is, every subset of the unit interval of the real number line is also a subset of 2^ω. The measure extension theorem then states that there is a single, unique measure $\tilde{\nu}$ on the unit interval of the real number line that is identical to ν on all subsets of the unit interval.

It is in this way that $?' \circ ?^{-1}$ should be understood as a measure on the unit interval, and, more precisely, a Lebesgue-Stieltjes measure. Since ? is continuous and invertible (one-to-one and onto), then $?'$ is a Lebesgue-Stieltjes measure as well. It is in this way that the use of the word "derivative" is not entirely inappropriate: Lebesgue-Stieltjes measures capture the essence of integration, and insofar as the integral is an anti-derivative, one can fairly casually talk about derivatives. In particular, later sections below will make use of the chain rule for derivatives $(?\circ f)' = (?'\circ f)\cdot f'$; the validity of the chain rule in this context works, because it can be seen to be a change-of-variable of a Lebesgue-Stieltjes integral.

The construction of $?'\circ ?^{-1}$ is such that it is also a measure (an outer measure!) on the even finer topology of 2^{ω}. All these various points will be implicit in what follows, where a more casual language will be adopted.

The product topology is also the setting for the so-called 'lattice models' of statistical physics. The next section explicitly develops the basic vocabulary of lattice models. The notion of fractal self-similarity can then be explicitly equated to shifts (translations) of the lattice. Equipped with this new vocabulary, it becomes possible to give an exact expression for the Minkowski measure μ to be expressed as an infinite product of Möbius functions.

3. Stern-Brocot convergent

The measure can be given a simple exact expression on dyadic intervals, through its relation to the Stern-Brocot tree. In this case, write

$$\Delta\left(\frac{b+a}{2}\right) = \frac{?(b)-?(a)}{b-a}$$

and consider the values $a = ?^{-1}(m/2^n)$ and $b = ?^{-1}((m+1)/2^n)$. Thes can be written recursively in terms of the Farey fractions in the Stern-Brocot tree; namely

$$?^{-1}\left(\frac{m}{2^n}\right) = \frac{p(m,n)}{q(m,n)}$$

where $p(m,n)/q(m,n)$ is the m'th fraction in the n'th row of the tree. These are given by the explicit recursion relation

$$p(m,n) = \begin{cases} p\left(\frac{m}{2}, n-1\right) & \text{if } m \text{ even} \\ p\left(\frac{m-1}{2}, n-1\right) + p\left(\frac{m+1}{2}, n-1\right) & \text{if } m \text{ odd} \end{cases}$$

and likewise for $q(m,n)$, the two differing only in the initial conditions:

$$\begin{aligned} p(0,0) &= 0 \\ p(1,0) &= 1 \\ q(0,0) &= 1 \\ q(1,0) &= 1 \end{aligned}$$

The partial convergents have the unit determinant; that is

$$p(m+1,n)q(m,n) - p(m,n)q(m+1,n) = 1$$

Combining all these together, one obtains the explicit expression

$$\Delta\left(\frac{b+a}{2}\right) = \frac{q(m+1,n)q(m,n)}{2^n}$$

Rather than using $(b+a)/2$as the midpoint, it is convenient to shift this slightly; this can be done because $?^{-1}$ is strictly increasing, and thus, midpoinnts are bounded by thier edges. Thus, since

$$?^{-1}\left(\frac{m}{2^n}\right) < ?^{-1}\left(\frac{2m+1}{2^{n+1}}\right) < ?^{-1}\left(\frac{m+1}{2^n}\right)$$

we may alter the definition of Δ and write

$$\Delta\left(?^{-1}\left(\frac{2m+1}{2^{n+1}}\right)\right)=\Delta\left(\frac{p(2m+1,n+1)}{q(2m+1,n+1)}\right)\equiv\Delta(m,n)=\frac{q(m+1,n)q(m,n)}{2^{n}}$$

which gives a fully recursive defintion for Δ valued on the rationals (since every rational occurs somewhere, uniquely, in the Stern-Brocot tree.

4. LATTICE MODELS

This section provides a review of the language of lattice models, setting up a minimal vocabulary needed for the remaining development. The use of lattice models is partly suggested by the developments above: the natural setting for probability theory is the product topology, and the natural setting for the product topology is the lattice model. The specific applicability of lattice models to the Minkowski measure is made clear in a later section.

Consider again the set of all possible (infinite length) strings in two letters. Each such string can be represented as $\sigma=(\sigma_0,\sigma_1,\sigma_2,\ldots)$ where $\sigma_k\in\{-1,+1\}$. These strings can be trivially equated to binary numbers, by writing the k'th binary digit as $b_k=(\sigma_k+1)/2$, so that for each configuration σ, one has the real number $0\leq x\leq 1$ given by

$$x(\sigma)=\sum_{k=0}^{\infty}b_k 2^{-(k+1)} \tag{4.1}$$

In physics, such strings are the setting for one-dimensional lattice models, the most famous of which is the Ising model. One considers each number k to represent a position in space, a lattice point. At each lattice point, one imagines that there is an atom, interacting with its neighbors, forming a perfect crystal. The goal of physics is to describe the macroscopic properties of this crystal given only microscopic details. There are two critical details that enter into all such models: the principle of maximum entropy, and translational invariance. These two principles, taken together, allow the Minkowski measure to be specified in an exact way.

The principle of maximum entropy states that all possible states of the model should be equally likely, subject to obeying any constraints that some or another average value or expectation value of a function of the lattice be fixed. There is a unique probability distribution that satisfies the principle of maximum entropy, it is known as the *Boltzmann distribution* or *Gibbs measure*. It is given by

$$\Pr(\sigma)=\frac{1}{Z(\Omega)}\,\exp\left(-\beta H(\sigma)\right) \tag{4.2}$$

where $\Pr(\sigma)$ denotes the probability of seeing the configuration σ. The normalizing factor $Z(\Omega)$ is known as the *partition function*, it exists to ensure that the sum of all probabilities totals unity. That is, one has

$$Z(\Omega)=\sum_{\sigma\in\Omega}\exp\left(-\beta H(\sigma)\right)$$

where Ω is the set of all possible configurations σ. The function $H(\sigma)$ is a real-valued function, conventionally called the 'energy' or 'Hamiltonian' in physics. The principle of maximum entropy interprets the energy as something whose expectation value is to be kept constant; that is, one adjusts the free parameter β in order to obtain a desired value for the expectation value

$$\langle H\rangle=\frac{1}{Z(\Omega)}\sum_{\sigma\in\Omega}H(\sigma)\exp\left(-\beta H(\sigma)\right)$$

The second principle commonly demanded in physics is that of translational invariance: that, because the crystalline lattice 'looks the same everywhere', all results should be independent of any specific position or location in the lattice. In particular, translational invariance implies that $H(\sigma)$ should have the same form as one slides around left or right on the lattice. In the present case, the lattice is one-sided, and so one can slide in only one direction. Defining the shift operator τ which acts on strings as $\tau(\sigma_0, \sigma_1, \sigma_2, \ldots) = (\sigma_1, \sigma_2, \sigma_3, \ldots)$, the principle of translation invariance states that we should only consider functions $H(\sigma)$ obeying the identity

$$H(\sigma) = H(\tau\sigma)$$

Unfortunately, for the present model, this cannot quite hold in the naive sense: the lattice is not double sided (with strings of symbols extending to infinity in both directions), but one-sided. The best one can hope for is to isolate the part that acts just on the first 'atom' in the left-most lattice location; call this part $V(\sigma)$, and subtract it from total: thus, instead, translational invariance suggests that, for the one-sided lattice, one should write

$$H(\sigma) = V(\sigma) + H(\tau\sigma) \tag{4.3}$$

That is, the total 'energy' contained in the lattice is equal to the energy contained in the first lattice position, plus the energy contained in the remainder of the lattice. That the energy should be additive (as opposed to being multiplicative, or combined in some other way) is consistent with its use as a log-linear probability in the Gibbs measure; that is, the use of the Gibbs measure, together with insistence that the model be translationally invariant, implies that the energy must be additive.

From this, it is is straight-forward to decompose $H(\sigma)$ into a sum:

$$H(\sigma) = \sum_{k=0}^{\infty} V\left(\tau^k \sigma\right) \tag{4.4}$$

What remains is the 'potential' $V(\sigma)$, which embodies the specifics of the problem at hand. The primary result of this note follows from the application of the above mechanics to the Minkowski measure: that is, to assume that the Minkowski measure is a Gibbs measure, and to solve for the potential $V(\sigma)$.

Before doing so, there is one more relationship to make note of: the shift operator τ, when looked at from the point of view as a function on real numbers, looks like multiplication-by-two. To be precise, one has the identity

$$2^k x(\sigma) - \left\lfloor 2^k x(\sigma) \right\rfloor = x\left(\tau^k \sigma\right)$$

where $\lfloor y \rfloor$ denotes the floor of y. The left-hand side of the above is a central ingredient of the construction of the Takagi or 'blancmange' curve; such a curve may be written as

$$\sum_{k=0}^{\infty} \lambda^k f\left(2^k x - \left\lfloor 2^k x \right\rfloor\right)$$

for various functions f of the unit interval. Such functions have many curious fractal self-similarity properties, and are explored in greater detail in [28, 27]. Writing $W = f \circ x$ shows that the Blancmange curve has the form

$$\sum_{k=0}^{\infty} \lambda^k W\left(\tau^k \sigma\right)$$

when considered on the lattice. The mapping given by eqn 4.1 is not just a mapping between strings and real numbers, it is a mapping from the product topology to the natural topology of the real number line. Thus, suddenly, many fractal and self-similarity phenomena are seen to be manifestations of simple structures in the product topology, which are then mapped to the natural topology.

5. SELF-SIMILARITY

Why should one even consider the application of lattice models to the Minkowski measure? The core justification is that the shift operator τ essentially acts as a multiplication-by-two, when considered on binary the numbers $x(\sigma)$ of eqn 4.1, together with the divide-by-two self-similarity relation in eqn 1.2. Thus, one posits that there is a probability distribution

$$\mathrm{Pr}(\sigma) = \left(?' \circ ?^{-1}\right)(x(\sigma)) \tag{5.1}$$

that is described by the Gibbs measure. The appearance here of the inverse $?^{-1}(x)$ is so as to be able to work 'in the same space' as where one started: that is, $?^{-1}(x)$ is a map that takes one from the space of dyadics to the space of Farey fractions, where an operation is performed that coincidentally takes one back to the space of dyadics. Seen in this way, eqn 5.1 is essentially the reciprocol of a Jacobian.

In 1991, Dieter Mayer hypothesized[22] that the distribution 5.1 was given by the Kac model[16], and specifically, by the Kac potential

$$V(x) = \begin{cases} -2x+1/2 & \text{for } 0 \le x \le 1/2 \\ 2x-3/2 & \text{for } 1/2 \le x \le 1 \end{cases} \tag{5.2}$$

while taking $\beta = 1$. The conjecture appears entirely reasonable, at least at the numerical level: inserting this into the Gibbs measure yields a graph that appears to be more-or-less equal the Question Mark, up to the level of numerical detail that can be easily achieved. However, Mayer does not provide a proof; rather, he intimates that it must be so.

In fact, numerical efforts suggest that there are many functions V that 'come close', and can serve as the basis for similar conjectures. Loosening the restriction of translational invariance, to allow Hamiltonians of the form $H(\sigma) = \sum_{k=0}^{\infty} \lambda^k V\left(\tau^k \sigma\right)$ seems to allow an ever richer collection of functions V that seem to yield good numerical approximations to the Question Mark. The Kac potential is simply a special case; as originally formulated, it is an interaction that decays exponentially with distance. Written in terms of interactions between lattice points, this exponential decay takes the form $H(\sigma) = \sum_{k=0}^{\infty} \lambda^k \sigma_0 \sigma_k$. Why there should be so many functions that seem to approximate the Question Mark is not immediately apparent; perhaps it is a statement that functions that seem 'close to' one another with respect to the natural topology of the unit interval have dramatically different form when considered in the product topology of the lattice model.

An exact solution has been promised; it is time to present it. To obtain the solution, one 'follows one's nose'. Starting with

$$\mathrm{Pr}(x) = \left(?' \circ ?^{-1}\right)(x)$$

one clearly has $\mathrm{Pr}(?(x)) = ?'(x)$. Applying the symmetry from eqn 1.2, one has that

$$\mathrm{Pr}\left(\frac{?(x)}{2}\right) = \mathrm{Pr}\left(?\left(\frac{x}{1+x}\right)\right) = ?'\left(\frac{x}{1+x}\right)$$

The division-by-two is to be recognized as the shift operator. That is, using translational symmetry 4.3 in the Gibbs measure 4.2 yields

$$\mathrm{Pr}(\sigma) = \exp(-V(\sigma))\,\mathrm{Pr}(\tau\sigma)$$

or equivalently,

$$\Pr\left(\frac{?(x)}{2}\right)=\exp\left(-V\left(\frac{?(x)}{2}\right)\right)\Pr\left(?(x)\right)$$

Back-substituting, one deduces

$$?'\left(\frac{x}{1+x}\right)=?'(x)\,\exp\left(-V\left(\frac{?(x)}{2}\right)\right)$$

On the other hand, one has

$$\frac{?'(x)}{2}=\frac{1}{(1+x)^2}?'\left(\frac{x}{1+x}\right) \tag{5.3}$$

which may be obtained by naively differentiating the relation 1.2. The casual use of differentiation here deserves some remark; it is justified by the same arguments that allow the “abuse of notation” of eqn 2.2; it holds as an identity involving the measure, a change of variable taken under the integral sign.

Continuing, the last result is plugged in to yield

$$\frac{(1+x)^2}{2}=\exp\left(-V\left(\frac{?(x)}{2}\right)\right)$$

which, after combining with eqn 1.2, is then easily solved for V to obtain

$$V(y)=\log 2+2\log\left(1-?^{-1}(y)\right)\qquad\text{for } 0\le y\le\frac{1}{2} \tag{5.4}$$

and $V(y)=V(1-y)$ for $1/2<x\le 1$. When graphed, this potential does appear to be vaguely tent-like, thus perhaps accounting for at least some of the success of the Kac potential to model the Question Mark.

To summarize the nature of this result: the potential 5.4, let us call it the ’Minkowski potential’, when used to formulate a lattice energy 4.4, gives the Minkowski measure 5.1 as a Gibbs measure 4.2. The next section of this note deals with the form that this measure when it is plugged back into this sequence of equations, culminating in the Gibbs measure.

6. A Product of Piece-wise Functions

Plugging the exact solution 5.4 into the Hamiltonian 4.4 and this in turn into the Gibbs measure 4.2 allows the derivative to be written as a product of piece-wise continuous functions. The result may be written as

$$?'(y)=\prod_{k=0}^{\infty}\frac{A'\circ A_k(y)}{2} \tag{6.1}$$

with the derivation given below.

Putting together the Hamiltonian 4.4 with the Gibbs measure 4.2 yields

$$\Pr(\sigma)=\frac{1}{Z(\Omega)}\prod_{k=0}^{\infty}\exp\left(-V\left(\tau^k\sigma\right)\right) \tag{6.2}$$

To obtain explicit expressions, it is convenient to perform the change-of-variable $x(\sigma)=?(y)$ so that eqn 5.1 becomes

$$\Pr(\sigma)=?'(y) \tag{6.3}$$

while eqn 5.4 contributes

$$\exp -V(\sigma) = \exp -V\left(\left(x^{-1}\circ ?\right)(y)\right) = \begin{cases} \frac{1}{2(1-y)^2} & \text{for } 0 \le y \le \frac{1}{2} \\ \frac{1}{2y^2} & \text{for } \frac{1}{2} \le y \le 1 \end{cases} \tag{6.4}$$

The remaining terms of the product need to be evaluated by considering the action of the shift operator τ. Under the change of variable $y = ?^{-1}(x(\sigma))$, this is given by

$$?^{-1}\left(x\left(\tau^k\sigma\right)\right) = ?^{-1}\left(\text{frac}\left(2^k ?(y)\right)\right)$$

where 'frac' is the fractional part of a number, that is, $\text{frac}(x) = x - \lfloor x \rfloor$. The general expression follows by iterating on the function

$$A(y) = ?^{-1}(\text{frac}(2?(y))) = \begin{cases} \frac{y}{1-y} & \text{for } 0 \le y \le \frac{1}{2} \\ \frac{2y-1}{y} & \text{for } \frac{1}{2} \le y \le 1 \end{cases} \tag{6.5}$$

Define the iterated function A_k so that

$$A_{k+1}(y) = A_k \circ A(y)$$

with $A_0(y) = y$ and $A_1(y) = A(y)$. Each of the A_k is made of 2^k Möbius transforms joined together; so, for example, the next one is

$$A_2(y) = \begin{cases} \frac{y}{1-2y} & \text{for } 0 \le y < \frac{1}{3} \\ \frac{3y-1}{y} & \text{for } \frac{1}{3} \le x < \frac{1}{2} \\ \frac{2y-1}{1-y} & \text{for } \frac{1}{2} \le x < \frac{2}{3} \\ \frac{3y-2}{2y-1} & \text{for } \frac{2}{3} \le x \le 1 \end{cases}$$

A graph of the first four is shown in figure 6.

The structure of the product can be elucidated by relating it to the structure of the dyadic monoid[27, 28]. A monoid is just a group, but without inverses; composing two elements of a monoid simply gives another element of the monoid. The dyadic monoid is a free monoid generated by two elements g and r, with with $r^2 = e$, and no constraints on g. Alternately, the dyadic monoid is the free monoid in two letters L and R; both presentations can be seen to correspond to a binary tree. In essence, the dyadic monoid consists of those moves that can navigate down an infinite binary tree.

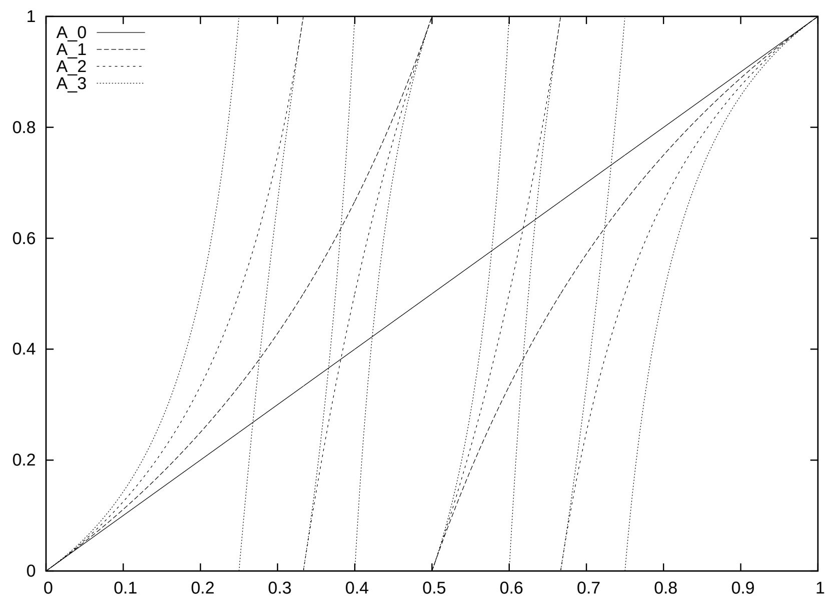
In this problem, the functions $g(y) = y/(1-y)$ and $r(y) = 1-y$ are the generators of the dyadic monoid. Explicitly, this is

$$A_{k+1}(y) = \begin{cases} (A_k \circ g)(y) = A_k\left(\frac{y}{1-y}\right) & \text{for } 0 \le y < \frac{1}{2} \\ (A_k \circ r \circ g \circ r)(y) = A_k\left(\frac{2y-1}{y}\right) & \text{for } \frac{1}{2} \le y \le 1 \end{cases}$$

Alternately, it can be convenient to work with the left $L = g$ and right $R = r \circ g \circ r$ transformations, with locations in the binary tree identified by navigating to the left or right, composing L and R.

The functions $A_k(y)$ reach their extremal values 0, 1 at exactly the Farey fractions from the k'th row of the Stern-Brocot tree. These may be obtained by following the movement of the endpoints 0,1 through successive iterations of L^{-1} and R^{-1}. These navigate "up" the tree; a point of occasional confusion can be avoided by keeping in mind that taking the inverse reverses the order of symbols, i.e. so that $(f \circ h)^{-1} = h^{-1} \circ f^{-1}$. For example, the third row of the Stern-Brocot tree is given by $\frac{1}{5} = L^{-1} \circ L^{-1'} \circ L^{-1}\left(\frac{1}{2}\right)$, $\frac{2}{7} = L^{-1} \circ L^{-1'} \circ R^{-1}\left(\frac{1}{2}\right)$, $\frac{3}{8} = L^{-1} \circ R^{-1'} \circ L^{-1}\left(\frac{1}{2}\right)$, $\frac{3}{7} = L^{-1} \circ R^{-1'} \circ R^{-1}\left(\frac{1}{2}\right)$, $\frac{4}{7} = R^{-1} \circ L^{-1'} \circ L^{-1}\left(\frac{1}{2}\right)$,

FIGURE 6.1. Sawteeth



and so on. Note that the order seems "reversed" from the natural order in which one might be tempted to apply the transformations L and R.

Of particular importance is that exponential of the potential, eqn 6.4, is just half the derivative of A:

$$\exp-\left(V\circ x^{-1}\circ ?\right)(y)=\frac{A'(y)}{2}$$

This is perhaps the most remarkable result of this development. That the Gibbs measure can be written as infinite product is no surprise. That repeated applications of the shift operator τ can be represented as the iteration of a function comes as no surprise either. That A is the proper form for the shift operator just follows from a study of the self-similarity properties of the Question Mark. However, the expression for the potential V, as written out in eqn 5.4, seems opaque, and without particular significance. Here, the potential is revealed to be "not just some arbitrary function", but related to the shift itself, and, more deeply, to the inverse of the transfer operator for the system. This is examined in greater detail below.

Several final remarks are in order. First, one has that the partition function $Z(\Omega)=1$. This is no accident, but was guaranteed by construction. Forcing the relationship 5.1, together with the fact that $?(1)=1$, ensured that all normalizing factors that might have appeared in $Z(\Omega)$ were in fact absorbed as an additive constant by V. Explicitly, this scaling leads to the constant $\log 2$ in eqn 5.4.

Caution should be used when evaluating eqn 6.1 numerically. If preparing a graph, or numerically evaluating its integral, the product is best terminated after $N\leq\log_2 h$ iterations, where h is the step size for the numerical integral. Using N larger than this bound will result in a very noisy sampling, resulting in numerical values that no longer resemble those of $?'$.

7. An Application

Perhaps the most immediate utility of the expression 6.1 is to provide insight into the question of "for what values of x is $?'(x)$ zero, and for what values is it infinite?"; such a question is explored in [9, 24]. So, for example: the A_k's can be seen to peel off binary digits, since $A_k(y) = ?^{-1}\left(\text{frac}\left(2^k ?(y)\right)\right)$. Thus, for any rational p/q, one has that $?(p/q)$ has a finite length N when expressed as a string of binary digits, and so one immediately has that $A_k(p/q) = 0$ when $k > N$: the infinite product terminates, and so of course, $?(p/q) = 0$.

More interesting is the case of y a quadratic irrational. In this case, $?(y)$ is a rational number, and $A_k(y)$, after a finite number M iterations, settles into a periodic orbit, of length $N = 2^K$ (of course: this is a basic result from the theory of Pellian equations). In this case, eqn 6.1 may be written as

$$?'(y) = \prod_{j=0}^{M-1} \frac{A' \circ A_k(y)}{2} \left[\prod_{k=0}^{N-1} \frac{A' \circ A_{k+M}(y)}{2}\right]^{\omega}$$

Clearly, the periodic part dominates. Writing

$$P(y) = \prod_{k=0}^{N-1} \frac{A' \circ A_{k+M}(y)}{2}$$

as the contribution from the periodic part, one clearly has that $?'(y) = 0$ whenever $P(y) < 1$, and $?'(y) = \infty$ when $P(y) > 1$. The accomplishment of [9] is to provide a precise description of these two cases, in terms of the continued fraction expansion of y. Thier work would appear to rule out any cases where $P(y) = 1$.

8. Modularity

The self-similarity transformation of the Minkowski measure resembles that of a modular form[4]. That is, given a general Möbius transform of the form $(ax+b)/(cx+d)$, with integers a,b,c,d and $ad - bc = 1$, the self-similarity of the Question Mark takes the form

$$?\left(\frac{ax+b}{cx+d}\right) = \frac{M}{2^N} + (-1)^Q \ \frac{?(x)}{2^K}$$

for some integers M,N,Q,K, depending on the specific values of a,b,c,d. Differentiating the above, one readily obtains

$$?'\left(\frac{ax+b}{cx+d}\right) = (cx+d)^2 \ \frac{?'(x)}{2^K}$$

which is almost exactly the transformation rule for a modular form of weight 2, and is spoiled only by the factor of 2^K. This suggests a conjecture: that there may be some holomorphic function, defined on the upper half-plane, for which the Minkowski measure is the limit of that function, the limit being taken a it's restriction to the real axis. Work in this direction has been given by Alkauskas, who notes that the generating function for the moments of the Question Mark is holomorphic in the upper half-plane[2], and gives Question Mark analogs for Maas wave forms, zeta functions and similar number-theoretic constructs[1].

This conjecture is best illustrated numerically. Take the product 6.1, terminating it after a finite number of terms; then graph the reciprocal. The resulting graph fairly resembles what one would obtain, if, for example, one graphed the Dirichlet eta function along a horizontal line, not far above the real axis. The Dirichlet eta is a prototypical example of a modular form; that it is also commonly written as an infinite product is also highly suggestive. These similarities bear further elucidation.

9. Generalized Shifts

Several generalizations of the Question Mark are possible. To properly define these, it is imperative to regain a clearer view of the lattice model methods. The ingredients to the above construction made use of a lattice Ω, whose states are encoded by σ, a lattice shift operator τ, and a function V on the lattice, to define a probability:

$$\Pr_V(\sigma) = \frac{1}{Z(\Omega)} \prod_{k=0}^{\infty} \exp\left(-V\left(\tau^k \sigma\right)\right)$$

A second important ingredient was to provide an explicit identification of lattice arrangements σ with real numbers y in the unit interval, given by $\sigma = x^{-1}(?(y))$. Putting these together lead to the form

$$P_f(y) = \frac{1}{Z(\Omega)} \prod_{k=0}^{\infty} f \circ A_k(y) \tag{9.1}$$

The appearance of A_k is explicitly governed by the mapping $\sigma = x^{-1}(?(y))$; that is, A_k is a representation of the shift operator τ^k, for this particular mapping of σ to real values. The function f is simply $\exp -V$, that is,

$$f(y) = \exp -V\left(x^{-1}(?(y))\right)$$

The partition function $Z(\Omega)$ is re-introduced as a reminder that the product definition only makes sense when f has been scaled appropriately; that is, f is arbitrary only up to a total scale factor; f exists only as an element of a projective space. The constraint is even stronger: inappropriately normalized f lead to the formal divergence $Z(\Omega) = \infty$; there is only one normalization for which $Z(\Omega)$ is finite, and that furthermore, by construction, one then has $Z(\Omega) = 1$.

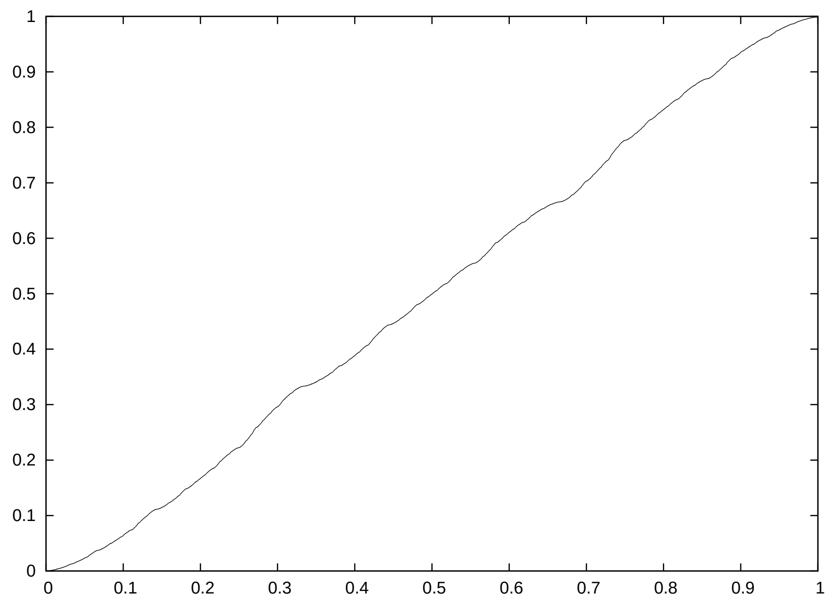
Two directions along which to generalize are now clear: one may consider general functions V, and one may consider different mappings from the lattice space Ω to the unit interval (which is compact, or to the positive real number line, which is not). The former is reviewed in a later section; lets treat the later case first.

The most widely used mapping is usually the much simpler one: $x = x(\sigma)$ from eqn 4.1, instead of the "twisted" mapping $\sigma = x^{-1}(?(y))$. The simpler mapping is commonly used to study the Bernoulli shift[7, 10, 14], an entry-point into the study of exactly solvable models in symbolic dynamics. By contrast, the "twisted" mapping allowed the Question mark to be properly formulated, thus suggesting that other maps can lead to other interesting objects. A twisted version of the Bernoulli Map will be presented in a later section.

The product form 6.1 provides the most concrete avenue immediately for generalizations. One need only to consider any differentiable, piece-wise function A and use it in the construction. For example, one may consider a 3-adic generalization of the Question Mark, which may be constructed from

$$B(y) = \begin{cases} \frac{2y}{1-y} & \text{for } 0 \le y < \frac{1}{3} \\ 3y-1 & \text{for } \frac{1}{3} \le y < \frac{2}{3} \\ \frac{3y-2}{y} & \text{for } \frac{2}{3} \le y \le 1 \end{cases} \tag{9.2}$$

FIGURE 9.1. The 3-adic Question Mark Function



and using B instead of A to generate the iterated function in the product, and $B'/3$ in place of $A'/2$ as the initial term. The resulting function, let's call it $?_3'$, can be integrated to define a 3-adic Question Mark $?_3$, which is shown in figure 9. Note that it requires no further normalization; one has $?_3(1) = 1$ simply by having divided B' by the number of branches of B. Visually, it is considerably less dramatic than the 2-adic Question Mark; hints of its self-similarity are visible but less apparent.

By construction, it has self-similarity, which can be immediately read off by considering B^{-1}, so, for example:

$$?_3\left(\frac{x}{2+x}\right) = \frac{?_3(x)}{3}$$

and

$$?_3\left(\frac{x+1}{3}\right) = \frac{1}{3} + \frac{?_3(x)}{3}$$

and

$$?_3\left(\frac{2}{3-x}\right) = \frac{2}{3} + \frac{?_3(x)}{3}$$

These self-similarity properties allow an explicit equation for $?_3$ to be written out, entirely analogous to Denjoy's explicit summation given in eqn 1.1, although now one must consider even and odd values of the terms a_k of the continued fraction distinctly. Whether, or how, the 3-adic, or, more generally, a p-adic Question Mark, inter-relate to results from Pellian equations, is not clear. Of some curiosity is whether the feature that caused Minkowski to present the Question Mark, namely, Lagrange's result relating periodic continued fractions to quadratic irrationals, might somehow be reflected, *e.g.* with cubic irrationals, and an appropriate 3-adic redefinition of periodicity in the continued fraction.

10. The Transfer Operator

The mapping from the lattice to the unit interval results in a collision of effects coming from different topologies: the product topology of the lattice, and the natural topology of the real number line. A standard way of studying this collision is by means of the transfer operator. The goal of this section is to introduce the transfer operator; the next section will show, in general terms, that the transfer operator is the push-forward of the shift operator.

Transfer operators, also commonly called Perron-Frobenius operators or Ruelle-Frobenius-Perron operators, as formulated by David Ruelle[xxx need ref] are a commonly-used tool for studying dynamical systems[7]. As the name "Perron-Frobenius" suggests, these are bounded operators, whose largest eigenvalue is equal to one; the existence of this eigenvalue is commonly justified by appealing to the Frobenius theorem, which states that bounded operators have a maximal eigenvalue. The origin of the name "transfer" comes from lattice-model physics, where the "transfer interaction" or the "transfer matrix" is studied[13, 11, 19]. In this context, the "transfer interaction" describes the effect of neighboring lattice sites on one-another; it can be seen to be a matrix when the number of interacting lattice sites, and the number states a single lattice point can be in, is finite. It becomes an operator when the interaction is infinite range, or when the set of states at a lattice site is no longer finite.

It is easiest to begin with a concrete definition. Consider a function $g:[0,1]\to[0,1]$, that is, a function mapping the unit interval of the real number line to itself. Upon iteration, the function may have fixed points or orbits of points. These orbits may be attractors or repellors, or may be neutral saddle points. The action of g may be ergodic or chaotic, strong-mixing or merely topologically mixing. In any case, the language used to discuss g is inherently based on either the point-set topology of the unit interval, or the "natural" topology on the unit interval, the topology of open sets.

A shift in perspective may be gained not by considering how g acts on points or open sets, but instead by considering how g acts on distributions on the unit interval. Intuitively, one might consider a dusting of points on the unit interval, with the local density given by $\rho(x)$ at point $x\in[0,1]$, and then consider how this dusting or density evolves upon iteration by g. This verbal description may be given form as

$$\rho'(y)=\int_0^1\delta\left(y-g(x)\right)\rho(x)\,dx \tag{10.1}$$

where $\rho'(y)$ is the new density at point $y=g(x)$ and δ is the Dirac delta function.

In this viewpoint, g becomes an operator that maps densities ρ to other densities ρ', or notationally,

$$\mathscr{L}_g\rho=\rho'$$

The operator $\mathscr{L}_g$ is the transfer operator or the Ruelle-Frobenius-Perron operator. It is not hard to see that it is a linear operator, in that

$$\mathscr{L}_g(a\rho_1+b\rho_2)=a\mathscr{L}_g\rho_1+b\mathscr{L}_g\rho_2$$

for constants a,b and densities ρ_1,ρ_2.

When the function g is differentiable, and doesn't have a vanishing derivative, the integral formulation of the transfer operator 10.1 can be rephrased in a more convenient form, as

$$\left[\mathscr{L}_g \rho\right](y) = \sum_{x:y=g(x)} \frac{\rho(x)}{|dg(x)/dx|} \tag{10.2}$$

where the sum is presumed to extend over at most a countable number of points. That is, when g is many-to-one, the sum is over the pre-images of the point y.

Both of the above definitions are mired in a language that implicitly assumes the real-number line, and its natural topology. Yet clearly, this is of limited utility; extracting the topology from the definition provides a clearer picture of the object itself.

11. The Push-Forward

In this section, it will be shown that the transfer operator is the push-forward of the shift operator; a theorem and a sequence of lemmas will be posed, that hold in general form. The point of this theorem is to disentangle the role of topology, and specifically, the role of measure theory, from the use of the shift operator. We begin with a general setting.

Consider a topological space X, and a field F over the reals $\mathbb{R}$. Here, F may be taken to be $\mathbb{R}$ itself, or $\mathbb{C}$ or some more general field over $\mathbb{R}$. The restriction of F to being a field over the reals is is required, so that it can be used in conjunction with a measure; measures are always real-valued.

One may then define the algebra of functions $\mathscr{F}(X)$ on X as the set of functions $f \in \mathscr{F}(X)$ such that $f : X \to F$. An algebra is a vector space endowed with multiplication between vectors. The space $\mathscr{F}(X)$ is a vector space, in that given two functions $f_1, f_2 \in \mathscr{F}(X)$, their linear combination $af_1 + bf_2$ is also an element of $\mathscr{F}(X)$; thus f_1 and f_2 may be interpreted to be the vectors of a vector space. Multiplication is the point-wise multiplication of function values; that is, the product $f_1 f_2$ is defined as the function $(f_1 f_2)(x) = f_1(x) \cdot f_2(x)$, and so $f_1 f_2$ is again an element of $\mathscr{F}(X)$. Since one clearly has $f_1 f_2 = f_2 f_1$, multiplication is commutative, and so $\mathscr{F}(X)$ is also a commutative ring.

The space $\mathscr{F}(X)$ may be endowed with a topology. The coarsest topology on $\mathscr{F}(X)$ is the *weak topology*, which is obtained by taking $\mathscr{F}(X)$ to be the space that is topological dual to X. As a vector space, $\mathscr{F}(X)$ may be endowed with a norm $\|f\|$. For example, one may take the norm to be the L^p-norm

$$\|f\|_p = \left(\int |f(x)|^p \, dx \right)^{1/p}$$

For $p = 2$, this norm converts the space $\mathscr{F}(X)$ into the Hilbert space of square-integrable functions on X. Other norms are possible, in which case $\mathscr{F}(X)$ has the structure of a Banach space rather than a Hilbert space.

Consider now a homomorphism of topological spaces $g : X \to Y$. This homomorphism induces the pullback $g^* : \mathscr{F}(Y) \to \mathscr{F}(X)$ on the algebra of functions, by mapping $f \mapsto g^*(f) = f \circ g$ so that $f \circ g : Y \to F$. The pullback is a linear operator, in that

$$g^*(af_1 + bf_2) = ag^*(f_1) + bg^*(f_2)$$

That the pullback is linear is easily demonstrated by considering how $g^* f$ acts at a point: $(g^* f)(x) = (f \circ g)(x) = f(g(x))$ and so the linearity of g^* on $af_1 + bf_2$ follows trivially.

One may construct an analogous mapping, but going in the opposite direction, called the push-forward: $g_* : \mathscr{F}(X) \to \mathscr{F}(Y)$. There are two ways of defining a push-forward. One way is to define it in terms of the sheaves of functions on subsets of X and Y. The sheaf-theoretic description is more or less insensitive to the ideas of measurability, whereas this is important to the definition of the transfer operator, as witnessed by the appearance of the

Jacobian determinant in equation 10.2. By contrast, the measure-theoretic push-forward captures this desirable aspect. It may be defined as follows.

One endows the spaces X and Y with sigma-algebras $(X,\mathscr{A})$ and $(Y,\mathscr{B})$, so that $\mathscr{A}$ is the set of subsets of X obeying the axioms of a sigma-algebra, and similarly for $\mathscr{B}$. A mapping $g : X \to Y$ is called "measurable" if, for all Borel sets $B \in \mathscr{B}$, one has the pre-image $g^{-1}(B) \in \mathscr{A}$ being a Borel set as well. Thus, a measurable mapping induces a push-forward on the sigma-algebras: that is, one has a push-forward $g_* : \mathscr{F}(\mathscr{A}) \to \mathscr{F}(\mathscr{B})$ given by $f \mapsto g_*(f) = f \circ g^{-1}$, which is defined by virtue of the measurability of g. The push-forward is a linear operator, in that

$$g_*(af_1+bf_2) = ag_*(f_1)+bg_*(f_2)$$

One regains the transfer operator as defined in equation 10.2 by considering the limiting behavior of the push-forward on progressively smaller sets. That is, one has

Theorem 1. *The transfer operator is the point-set topology limit of the measure-theoretic push-forward.*

Proof. The proof that follows is informal, so as to keep it simple. It is aimed mostly at articulating the language and terminology of measure theory. The result is none-the-less rigorous, if taken within the confines of the definitions presented.

Introduce a measure $\mu : \mathscr{A} \to \mathbb{R}^+$ and analogously $\nu : \mathscr{B} \to \mathbb{R}^+$. The mapping g is measure-preserving if ν is a push-forward of μ, that is, if $\nu = g_*\mu = \mu \circ g^{-1}$. The measure is used to rigorously define integration on X and Y. Elements of $\mathscr{F}(\mathscr{A})$ can be informally understood to be integrals, in that $f(A)$ for $A \in \mathscr{A}$ may be understood as

$$f(A) = \int_A \tilde{f}(z)d\mu(z) = \int_A \tilde{f}(z)\left|\mu'(z)\right|dz$$

where $|\mu'(x)|$ is to be understood as the Jacobean determinant at a point $x \in X$. Here, $\tilde{f}$ can be understood to be a function that is being integrated over the set A, whose integral is denoted by $f(A)$. The value of $\tilde{f}$ at a point $x \in X$ can be obtained by means of a limit. One considers a sequence of $A \in \mathscr{A}$, each successively smaller than the last, each containing the point x. One then has

$$\varinjlim_{A\ni x} \frac{f(A)}{\mu(A)} = \tilde{f}(x)$$

which can be intuitively proved by considering A so small that $\tilde{f}$ is approximately constant over A:

$$f(A) = \int_A \tilde{f}(z)d\mu(z) \approx \tilde{f}(x)\int_A d\mu = \tilde{f}(x)\mu(A)$$

To perform the analogous limit for the push-forward, one must consider a point $y \in Y$ and sets $B \in \mathscr{B}$ containing y. In what follows, it is now assumed that $g : X \to Y$ is a multi-sheeted countable covering of Y by X. By this it is meant that for any y that is not a branch-point, there is a nice neighborhood of y such that its pre-image consists of the union of an at most countable number of pair-wise disjoint sets. That is, for y not a branch point, and for $B \ni y$ sufficiently small, one may write

$$g^{-1}(B) = A_1 \cup A_2 \cup \cdots = \bigcup_{j=1}^{k} A_j$$

where k is either finite or stands for ∞, and where $A_i \cap A_j = \varnothing$ for all $i \neq j$. At branch points, such a decomposition may not be possible. The axiom of sigma-additivity guarantees that such multi-sheeted covers behave just the way one expects integrals to behave: in other words, one has

$$\mu\left(g^{-1}(B)\right)=\mu\left(\bigcup_{j=1}^{k}A_j\right)=\sum_{j=1}^{k}\mu\left(A_j\right)$$

whenever the collection of A_j are pair-wise disjoint. Similarly, in order to have the elements $f\in\mathscr{F}(\mathscr{A})$ behave as one expects integrals to behave, one must restrict $\mathscr{F}(\mathscr{A})$ to contain only sigma-additive functions as well, so that

$$f\left(g^{-1}(B)\right)=f\left(\bigcup_{j=1}^{k}A_j\right)=\sum_{j=1}^{k}f\left(A_j\right)$$

As the set B is taken to be smaller and smaller, the sets A_j will become smaller as well. Denote by x_j the corresponding limit point of each A_j, so that $g(x_j)=y$ and the pre-image of y consists of these points: $g^{-1}(y)=\left\{x_1,x_2,\cdots\,\middle|\,g(x_j)=y\right\}$. One now combines these provisions to write

$$\begin{aligned}\left[g_*\tilde{f}\right](y) &= \lim_{\overrightarrow{B\ni y}}\left[\frac{(g_*f)(B)}{\nu(B)}\right]\\ &= \lim_{\overrightarrow{B\ni y}}\left[\frac{\left(f\circ g^{-1}\right)(B)}{\nu(B)}\right]\\ &= \lim_{\overrightarrow{A_j\ni g^{-1}(y)}}\frac{f\left(A_1\cup A_2\cup\cdots\right)}{\nu(B)}\\ &= \lim_{\overrightarrow{A_j\ni g^{-1}(y)}}\frac{\sum_{j=1}^{k}f\left(A_j\right)}{\nu(B)}\\ &= \sum_{j=1}^{k}\tilde{f}\left(x_j\right)\lim_{\overrightarrow{A_j\ni x_j}}\frac{\mu\left(A_j\right)}{\nu(B)} \qquad (11.1)\end{aligned}$$

The limit in the last line of this sequence of manipulations may be interpreted in two ways, depending on whether one wants to define the measure ν on Y to be the push-forward of μ, or not. If one does take it to be the push-forward, so that $\nu=g_*\mu$, then one has

$$\lim_{\overrightarrow{A_j\ni x_j}}\frac{\mu\left(A_j\right)}{g_*\mu(B)}=\frac{1}{\left|g'\left(x_j\right)\right|}$$

where $\left|g'\left(x_j\right)\right|$ is the Jacobian determinant of g at x_j. This last is a standard result of measure theory, and can be intuitively proved by noting that $g\left(A_j\right)=B$, so that

$$\nu(B)=\int_{A_j}g'(z)\,d\mu(z)\approx g'\left(x_j\right)\mu\left(A_j\right)$$

for "small enough" B. Assembling this with the previous result, one has

$$\left[g_*\tilde{f}\right](y)=\sum_{x_j\in g^{-1}(y)}\frac{\tilde{f}\left(x_j\right)}{\left|g'\left(x_j\right)\right|} \qquad (11.2)$$

which may be easily recognized as equation 10.2. This concludes the proof of the theorem, that the transfer operator is just the point-set topology limit of the push-forward. $\square$

In simplistic terms, the push-forward can be thought of as a kind of change-of-variable. Thus, one should not be surprised by the following lemma, which should be recognizable as the Jacobian, from basic calculus.

Lemma 2. *(Jacobian) One has*

$$\sum_{x_j \in g^{-1}(y)} \frac{1}{\left|g'(x_j)\right|} = 1$$

Proof. This follows by taking the limit $\overrightarrow{A_j \ni x_j}$ of

$$\frac{\mu(A_j)}{g_*\mu(B)} = \frac{\mu(A_j)}{\sum_{i=1}^k \mu(A_i)}$$

and then summing over j. □

Corollary 3. *The uniform distribution (*i.e. *the measure) is an eigenvector of the transfer operator, associated with the eigenvalue one.*

Proof. This may be proved in two ways. From the viewpoint of point-sets, one simply takes $\tilde{f} = \text{const.}$ in equation 11.2, and applies the lemma above. From the viewpoint of the sigma-algebra, this is nothing more than a rephrasing of the starting point, that $\nu = g_*\mu$, and then taking the space $Y = X$, so that the push-forward induced by $g : X \to X$ is a measure-preserving map: $g_*\mu = \mu$. □

The last corollary is more enlightening when it is turned on its side; it implies two well-known theorems, which follow easily in this framework.

Corollary 4. *(Ruelle-Perron-Frobenius theorem). All transfer operators are continuous, compact, bounded operators; furthermore, they are isometries of Banach spaces.*

Proof. This theorem is of course just the Frobenius-Perron theorem, recast in the context of measure theory. By definition, the measures have unit norm: that is, $\|\mu\|_1 = 1$ and $\|\nu\|_1 = 1$. This is nothing more than the statement that the spaces X and Y are measurable: the total volume of X and Y is, by definition, one. Since $\nu = g_*\mu$, we have $\|g_*\mu\|_1 = 1$, and this holds for all possible measures $\mu \in \mathscr{F}(X)$.

Recall the definition of a bounded operator. Given a linear map $T : U \to V$ between Banach spaces $U$and V, then T is bounded if there exists a constant $C < \infty$ such that $\|Tu\|_V \le C\|u\|_U$for all $u \in U$. But this is exactly the case above, with $T = g_*$, and $U = \mathscr{F}(X)$, $V = \mathscr{F}(Y)$, and $C = 1$. The norm of a bounded operator is conventionally defined as

$$\|T\| = \sup_{u \neq 0} \frac{\|Tu\|_V}{\|u\|_U} = \sup_{\|u\|_U \le 1} \|Tu\|_V$$

and so we have the norm of g_* being $\|g_*\| = 1$. That g_* is an isometry follows trivially from $\|g_*\mu\|_1 = \|\nu\|_1$ and that g_* is linear. □

The corollary 3 can also be treated as a corollary to the Perron-Frobenius theorem: namely, that there is at least one vector that corresponds to the maximum eigenvalue of g_*. This eigenvector is in fact the Haar measure, as the next theorem shows.

Theorem 5. *(Haar measure) For any homomorphism $g : X \to X$, one may find a measure μ such that $g_*\mu = \mu$; that is, every homomorphism g of X induces a measure μ on X such that g is a measure-preserving map. If g is ergodic, then the measure is unique.*

Proof. By definition, μ is a fixed point of g_*. The fixed point exists because $g_*$is a bounded operator, and the space of measures is compact, and so a bounded operator on a compact space will have a fixed point. The existence of the fixed point is given by the Markov-Kakutani theorem[8, p 456]. The Markov-Kakutani theorem also provides the uniqueness condition: if there are other push-forwards h_* that commute with g_*, then each such push-forward will also have a fixed point. The goal is then to show that when g is ergodic, there are no other functions h that commute with g. But this follows from the definition of ergodicity: when g is ergodic, there are no invariant subspaces, and the orbit of g is the whole space. As there are no invariant subspaces, there are no operators that can map between these subspaces, *i.e.* there are no other commuting operators. $\square$

A peculiar special case is worth mentioning: if g is not ergodic on the whole space, then typically one has that the orbit of g splits or foliates the measure space into a bunch of pairwise disjoint leaves, with g being ergodic on each leaf. The Markov-Kakutani theorem then implies that there is a distinct fixed point μ in each leaf, and that there is a mapping that takes μ in one leaf to that in another.

In the language of dynamical systems, the push-forward g_* is commonly written as $\mathscr{L}_g$, so that one has

$$g_* = \mathscr{L}_g : \mathscr{F}(X) \to \mathscr{F}(X)$$

now being called the transfer operator or the Ruelle-Frobenius-Perron operator.

In the language of physics, the fixed point μ is called the "ground state" of a system. When it is unique, then the ground state is not degenerate; when it is not unique, then the ground state is said to be degenerate. The operator g_* is the time-evolution operator of the system; it shows how physical fields $f \in \mathscr{F}(X)$ over a space X evolve over time. When F is the complex numbers $\mathbb{C}$, the fact that $\|g_*\| = 1$ is essentially a way of stating that time-evolution is unitary; the Frobenius-Perron operator is the unitary time-evolution operator of the system. What is called "second quantization" in physics should be interpreted as the fitting of the space $\mathscr{F}(X)$ with a set of basis vectors, together with a formulation of g_* in terms of that basis.

12. Grounding

To make sense of the discussion above, it is best to ground the symbols in concrete terms. For the most part, this paper is concerned either with the space $X = \Omega$, the space of strings in two letters, or the spaces X that are somehow isomorphic to Ω. This space is a (semi-)infinite product of $\mathbb{Z}_2 = \{0, 1\}$; elements $\sigma \in \Omega$ of this space were values $\sigma = (\sigma_0, \sigma_1, \cdots)$ with each $\sigma_k \in \mathbb{Z}_2$. The function g is to be equated with the shift operator τ; that is, when $X = \Omega$, then $g = \tau$ and $g : X \to X$ is just $\tau : \Omega \to \Omega$ acting such that $\tau(\sigma_0, \sigma_1, \sigma_2, \cdots) = (\sigma_1, \sigma_2, \cdots)$. Curiously, the structure of the space Ω can be understood to be a binary tree: starting at σ_0 one can branch to the left or to the right, to get to σ_1, and so on.

However, a product space such as this also allows the language of probability theory come into play, and this is where things get interesting. Product spaces are the natural setting for probability: the element $\sigma \in \Omega$ is called a "random variable", and successive values σ_k of σ are "measurements" of the random variable. This is also the manner in which measure theory gets its foot in the door.

The mappings $x : \Omega \to \mathbb{R}$ given by eqn 4.1 is now seen to be an example of an element of the algebra $\mathscr{F}(\Omega)$ introduced in the previous section. The mapping $y : \Omega \to \mathbb{R}$ given by $\sigma \mapsto y = ?^{-1}(x(\sigma))$ is another example of an element of $\mathscr{F}(\Omega)$. Both the maps x and y

are invertible on the closed unit interval, establishing a $1-1$ mapping between Ω and the closed unit interval. Consider, then, the function

$$b = x \circ \tau \circ x^{-1}$$

which maps the unit interval into itself. Let $w \in [0,1]$ be a point in the unit interval; then it is not hard to see that $b(w) = 2w \bmod 1$. In the literature, this function is commonly called the Bernoulli shift; its corresponding random process is known as the Bernoulli process. It is well-known to be ergodic. The other function that was considered here was

$$A = y \circ \tau \circ y^{-1}$$

Its action on the unit interval has already been given by eqn 6.5. In a previous section, we have already considered generalizations of A, that is, other functions $B : [0,1] \to [0,1]$, such as, for example, eqn 9.2.

Each of these maps b, A, B can be taken as an example of a map $g : X \to X$, now taking X to the unit interval. By the previous theorem, each such map g induces a push-forward $g_* = \mathscr{L}_g$, and this push-forward then induces an invariant measure μ_g on X. For the Bernoulli map b, the push-forward $\mathscr{L}_b$ is introduced in [10, 14], and a detailed study of its eigenvectors is given in [7]. The invariant measure μ_b is well-known to be the identity on the unit interval. This just means that b considered as an ergodic process, is uniformly distributed on the unit interval.

The invariant measure μ_A has already been established in eqn 6.3 as being the Minkowski measure; that is, $\mu_A = ?'$. A previous section already gave specific mechanics for how to take a general, differentiable onto-map B and define its associated measure μ_B as the generalization of eqn 6.1. The next section introduces the transfer operator $\mathscr{L}_A$ corresponding to A.

13. FLIGHTINESS

To counterbalance the grounding, a brief bit of flightiness as to this process is offered up. The map $g : X \to X$, can be understood to give the (discrete) time-evolution of a dynamical system. It is curious to ponder that iterating on g can be thought of as the (semi-)infinite chain

$$X \xrightarrow{g} X \xrightarrow{g} X \xrightarrow{g} X \xrightarrow{g} \cdots$$

and so any particular g, together with an initial state x_0, can be interpreted as selecting a particular sequence $(x_0, x_1, x_2, \ldots)$ out of the (semi-)infinite product space $\Omega = X \times X \times X \times \cdots$. The set membership function now defines a measure on Ω: sets that contain sequences $(x_0, x_1, x_2, \ldots)$ for which $x_{k+1} = g(x_k)$ will have a non-zero measure, all others will have a measure of zero.

This now begs the question: what sort of operators might have this measure as their invariant measure? Which of these operators might be push-forwards? What would they be a push-forward of? Last but not least, all of the spaces under discussion are metric spaces, and so it would be curious to know "what else is close by". These questions are not entirely idle. The product space Ω can be seen to be "space-time"; by analogy to the binary tree, the structure of this space-time is once-again seen to be branching; infinitely branching in this case. But such branching is already seen in physics: this is nothing other than the many-worlds interpretation of quantum mechanics, where, at each moment of time, the universe is seen to split into an "infinite number of copies of itself". The many-worlds branching structure is not well-explored or well-understood, although it can illuminate problems of quantum measurement and wave-function collapse. One such curious problem is the so-called "Mott problem". The question that is posed by the Mott problem is "why does the spherically symmetric wave function of an emitted alpha-ray always leave a straight-line track in a Wilson cloud chamber?" The answer was given by Heisenberg and Mott in 1929; it can be re-interpreted in the context of the current formulation as a statement that straight-line tracks have a significant measure in this branching space-time, as they all correspond to sets that contain the point that is the spherical wave-function, and no other sets do. But this is just an impressionistic daydream, lacking rigor; the hope is that it motivates further study.

14. The Twisted Bernoulli Operator

We know consider the transfer operator or push-forward of the mapping A. It resembles the Bernoulli shift, but is twisted in a curious way; it is a continued-fraction analog of the Bernoulli shift. It is straight-forward to write it down; it is very simply

$$[\mathscr{L}_A f](y) = \frac{1}{(1+y)^2} f\left(\frac{y}{1+y}\right) + \frac{1}{(2-y)^2} f\left(\frac{1}{2-y}\right)$$

This expression is easily obtained by starting with eqn 10.2 and substituting in A from eqn 6.5.

Of course, the Minkowski measure is an eigenfunction of the twisted Bernoulli shift; that is, one has

$$\mathscr{L}_A ?' = ?'$$

This can be seen in several ways: The Minkowski measure was constructed as the Haar measure induced by A, and so by theorem 5, of course this identity must hold. But it can also be validated by more pedestrian methods: simply plug in the measure, and turn the crank, making use of the self-similarity relation 5.3 as needed.

Other eigenvalues of $\mathscr{L}_A$ are readily found. Consider the functions P_f as previously defined in 9.1. The action of the twisted Bernoulli shift on P_f is readily computed, and is given by

$$\mathscr{L}_A P_f = P_f \cdot \mathscr{L}_A f$$

We've already observed that when $f = A'/2$, that one has

$$\mathscr{L}_A \frac{A'}{2} = 1$$

so that $P_{A'/2} = ?'$. Another, linearly independent function which satisfies $\mathscr{L}_A f = 1$ is

$$C(y) = \begin{cases} \frac{1}{3(1-y)^3} & \text{for } 0 \le y \le \frac{1}{2} \\ \frac{1}{3y^3} & \text{for } \frac{1}{2} \le y \le 1 \end{cases}$$

that is, $\mathscr{L}_A C = 1$. The collection of all f which satisfy $\mathscr{L}_A f = 1$ is large, but is easily specified. Let

$$f(y) = \begin{cases} f_0(y) & \text{for } 0 \le y \le \frac{1}{2} \\ f_1(y) & \text{for } \frac{1}{2} \le y \le 1 \end{cases}$$

then, given any, arbitrary f_0, solving for $\mathscr{L}_A f = 1$ results in

$$f_1(y) = \frac{1}{y^2} - \frac{1}{(3y-1)^2} f_0\left(\frac{2x-1}{3x-1}\right)$$

Such f are properly scaled; that is, $Z(\Omega) = 1$. Because of the projective relationship between the functions f and the normalization $Z(\Omega)$, it appears that this technique will result in only in eigenfunctions P_f with eigenvalue 1.

Keep in mind that while $P_{f+h} \neq P_f + P_h$, such a relationship does hold in the logarithmic sense: $P_{fh} = P_f P_h$, thus P can be construed to be a (logarithmic) homomorphism of Banach spaces.

15. THE GAUSS-KUZMIN-WIRSING OPERATOR

The Gauss-Kuzmin-Wirsing (GKW) operator is the transfer operator of the Gauss map $h(x) = 1/x - \lfloor 1/x \rfloor$. Denoting this operator as $\mathscr{L}_G$, it acts on a function f as

$$[\mathscr{L}_G f](x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} f\left(\frac{1}{x+n}\right) \tag{15.1}$$

It has a well-known eigenfunction, noted by Gauss, corresponding to eigenvalue one, given by $1/(1+y)$. It is also relatively straightforward to observe that the Minkowski measure is also an eigenfunction, also corresponding to eigenvalue one. This is readily seen by differentiating the Question Mark: one has

$$?\left(\frac{1}{n+x}\right) = \frac{1}{2^{n-1}} - \frac{?(x)}{2^n}$$

and so also

$$?'\left(\frac{1}{n+x}\right) = \frac{(x+n)^2}{2^n} ?'(x)$$

This can be directly inserted into eqn 15.1 to very easily find that

$$\mathscr{L}_G ?' = ?'$$

This result immediately suggests that the lattice model methods can be applied to find other eigenfunctions. (The 3-adic Question mark, at least as defined above, does not have this property, suggesting, perhaps, that a different 3-adic Question Mark should be formulated).

The GKW operator is tied by various identities to the twisted Bernoulli operator $\mathscr{L}_A$. These are explicitly given by [15]; they can be crudely summarized by saying that these identities are the operator analogues of Denjoy's explicit summation 1.1. In these rough terms, the Bernoulli shift chops off one digit of a binary number at a time; these may be in turn re-assembled to form the GKW operator.

The action of the GKW operator $\mathscr{L}_G$ on P_f is not at all tidy; the result that $\mathscr{L}_G ?' = ?'$ appears to be entirely due to the special form of $f = A'/2$. This is perhaps no surprise.

Other fractal eigenvectors for the GKW can be constructed by a technique entirely parallel to the construction of the P_f; one only has to use a different lattice model. For this model, construct

$$R_f(y) = \frac{1}{Z(\Omega)} \prod_{k=0}^{\infty} f \circ h_k(y)$$

where $h_{k+1} = h_k \circ h$ is function iteration, and $h(y) = \mathrm{frac}(1/y)$ is the Gauss map, as before. This multi-branched h has the property that

$$h\left(\frac{1}{x+n}\right) = x$$

for all integer values of $n \geq 1$. Applying the GKW operator to R_f results in

$$\mathscr{L}_G R_f = R_f \mathscr{L}_G f$$

in analogy to the twisted Bernoulli operator. As before, functions f for which $\mathscr{L}_G f = 1$ also implies that $Z(\Omega) = 1$. The means for finding solutions to $\mathscr{L}_G f = 1$ proceeds analogously to the twisted-Bernoulli case. Thus, let

$$f(y) = \begin{cases} f_1(y) & \text{for } \frac{1}{2} < y \leq 1 \\ \cdots & \cdots \\ f_n(y) & \text{for } \frac{1}{n+1} < y \leq \frac{1}{n} \\ \cdots & \cdots \end{cases}$$

The equation $\mathscr{L}_G f = 1$ establishes only a single constraint between these f_n's: all but one may be freely chosen. As an example, consider

$$f_n(y) = \frac{a_n}{y^2}$$

Then the resulting f satisfies $\mathscr{L}_G f = 1$ provided that the constants a_n satisfy

$$\sum_{n=1}^{\infty} a_n = 1$$

By taking $a_n = 2^{-n}$, one regains the Question Mark function; that is, $R_f = ?'$ for this case. Quick numerical work shows that other geometric series return commonly-seen "Question-Mark-like" distributions, while arithmetic series, such as $a_n = 6/(\pi n)^2$, yeild more unusual shapes.

This construction shows that the space of fractal eigenfunctions of the GKW operator is very large; but, as before, all of these eigenfunctions appear to be non-decaying, corresponding to eigenvalue 1.

16. Conclusion

The principal results of this note are the statistical distribution of equation 2.1, which allows numerical explorations of the binary trees to be explicitly equated to the Minkowski measure, and equation 6.1, which shows that the seemingly intractable Minkowski measure can be built up out of analytic functions in a regular way. A sufficient amount of formal theory is developed to show how to root this measure in a foundational apparatus, thus enabling various basic manipulations on it to be safely performed. Of these, the most notable was to expose the transfer operator as a push-forward on a Banach space, leading to an interpretation of the Minkowski measure as a invariant measure, a Haar measure, induced by a dynamical map.

An important theme is the tension between the natural topology on the real number line, and the much finer 'weak topology' on the product space 2^ω. It should be clear that the fine topology is the source for a broad variety for fractal, self-similar phenomena. The use of the measure extension theorem allows these fine-grained fractal phenomena to manifest themselves onto the natural topology; the Minkowski measure serves as a non-trivial yet still simple example of such a manifestation.

However, various open problems remain. There have now been many generalizations of the Question Mark proposed; besides that of Denjoy, and those of this note, some additional ones are given in [3]. How are all of these related? How are they best enumerated? Is there some characteristic labelling that best distinguishes each form? Insofar as these variants do occasionally manifest in practical problems, it is useful to have a coherent naming system to describe them.

It is noted in section 5 that there seem to be many different functions V that generate functions Q that seem close to the Question Mark; in some cases, these appear to be arbitrarily close, when studied numerically. Why is this? Is there an actual degeneracy, with different V's generating the same function Q? If not, then the metric on the Banach space of different V's must be wildly folded to yeild close-by Q's: the metrics are not "compatible", or presumably discontinuous in some way. The structure of the mapping from the space of V's to the space of Q's is opaque, and would benefit from clarification.

Section 9 considered a 3-adic form of the Question Mark. Its construction is somewhat *ad hoc*, and it's properties are less than pretty: for example, it is not an eigenfunction of the GKW operator. Is there another construction that would be an eigenfunction? If not, why not? If so, what would the correct p-adic generalization be? Let's assume that unambiguous p-adic generalizations can be written. What are thier connections to number theory? That is, $?(x)$ maps rationals to quadratic surds; is there an analogous statement for such $?_p(x)$?

The most important open problem surrounds the observation that transfer operators have a discrete spectrum, when the space of functions is restricted to a much smaller set, for instance, when it is restricted to the set of square-integrable functions, or to the set of polynomials. In some cases, this spectrum can be obtained exactly; this has been done for the case of the Bernoulli operator [10, 14, 7]. More generally, obtaining this spectrum seems almost intractable; here, the prototypical example is the GKW operator. Of course, both the eigenvalues and the eigenvectors of the GKW operator can be computed numerically; however, there is no articulation or framework within which to talk about them and to manipulate them; they remain opaque. They seem to be mere holomorphic functions (by definition!) and oscillatory, of course; but there is not yet a way of writing down their power series directly.

Thus, the open problem is to develop a technical framework that will identify the discrete eigenvalue spectrum of a transfer operator, and to directly provide expressions for the holomorphic functions which are their eigenvectors. Such a framework would be of broad utility.

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